

Whitney towers, Gropes and Casson-Gordon style invariants of links

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Abstract. In this paper, we prove a conjecture of Friedl and Powell that their Casson-Gordon type invariant of 2-component link with linking number one is actually an obstruction to being height 3.5 Whitney tower/grope concordant to the Hopf Link. The proof employs the notion of solvable cobordism of 3-manifolds with boundary, which was introduced by Cha. We also prove that the Blanchfield form and the Alexander polynomial of links in S^3 give obstructions to height 3 Whitney tower/grope concordance. This generalizes the results of Hillman and Kawauchi.

1. Introduction

In the study of topological knot concordance, various invariants were introduced in seminal papers including [Lev69], [CG78, CG86], and [COT03]. All of these invariants can be extracted from the 0-surgery manifolds of knots. Influenced by these works, the link slicing problem has been studied extensively using various covers of the 0-surgery manifolds of links. For example, [Har08], [CHL08], and [Hor10] used Cheeger-Gromov ρ -invariants from PTFA (poly-torsion-free-abelian) covers. In [Cha10] and [Cha09], Hirzebruch type invariants from iterated prime power fold covers are defined and used.

In general link concordance problems, it is known that zero surgery manifolds do not reveal full information. For example, for 2-component links with linking number one, aforementioned invariants automatically vanish. In fact, those invariants are obtained from solvable covers of zero surgery manifolds. For 2-component links with linking number one, there are no non-trivial solvable covers of the zero surgery manifolds (and consequently aforementioned invariants vanish) because they have perfect fundamental groups. Also, there is an in-depth study which presents related results about link concordance versus zero surgery homology cobordism, see [CP].

Recently, for 2-component links with linking number one, S. Friedl and M. Powell [FP14] introduced a Casson-Gordon style metabelian invariant $\tau(L, \chi)$ by considering another closed 3-manifold obtained from the link exterior. Also, they found new 2-component links with linking number one which are not concordant to the Hopf link. The aim of this paper is to give a better understanding of $\tau(L, \chi)$ in the context of symmetric Whitney towers and gropes in dimension 4.

Friedl-Powell invariant $\tau(L, \chi)$

To describe our main result, we briefly summarize the construction and main result in [FP14]. (For more details, see Section 4.)

Let L be an ordered, oriented 2-component link with linking number 1 in S^3 and H be the Hopf link. Define M_L to be the closed 3-manifold obtained by gluing the exteriors of L and H along their boundary, identifying the meridians of corresponding components. For a prime p , choose a homomorphism $\varphi: H_1(M_L; \mathbb{Z}) \rightarrow \mathbb{Z}/p^i \times \mathbb{Z}/p^j$ which sends two meridians of L to the standard basis $(1, 0)$ and $(0, 1)$, respectively. Let $M_L^\varphi \rightarrow M_L$ be the p^{i+j} -fold covering induced by φ . For a prime q and a character $\chi: H_1(M_L^\varphi; \mathbb{Z}) \rightarrow \mathbb{Z}/q^k$, Friedl and Powell define an invariant

$$\tau(L, \chi) \in L^0(\mathbb{C}(\mathcal{H})) \otimes_{\mathbb{Z}} \mathbb{Z}[1/q]$$

in [FP14, Section 3.2] (see also our Definition 4.1). Here, $\mathcal{H} = \mathbb{Z}^3$, $\mathbb{C}(\mathcal{H})$ is the quotient field of the group ring $\mathbb{C}[\mathcal{H}]$, and $L^0(\mathbb{C}(\mathcal{H}))$ is the Witt group of finite dimensional non-singular sesquilinear forms over $\mathbb{C}(\mathcal{H})$. The main result of [FP14] essentially says that if L is concordant to H , then $\tau(L, \chi)$ vanishes [FP14, Theorem 3.5]. For a precise definition of the vanishing of $\tau(L, \chi)$, see Definition 4.2. We omit the precise statement here because we need to discuss some technicality including the choice of a metabolizer of the linking form.

Symmetric Whitney tower/grope concordance and $\tau(L, \chi)$

The symmetric Whitney towers and gropes are approximations of embedded surfaces which play the central role in the study of topological 4-manifolds. For example, a special kind of grope with caps gives a topologically embedded disk in the disk embedding theorem of [FQ90]. Also, using symmetric Whitney towers and gropes, T. Cochran, K. Orr, and P. Teichner developed the filtration theory of the knot concordance group [COT03]. It turns out that the structure of this filtration theory is extremely rich (for example, see [COT04], [CT07], [CHL09], [CHL11], and [Cha13]). For links, we are mainly interested in two equivalence relations, *height h Whitney tower concordance* and *height h grope concordance*. (For precise definitions, see [Cha14, Definitions 2.12, 2.15].)

We remark that J. Conant, R. Schneiderman, and Teichner developed another interesting filtration theory using coarser notion called order n Whitney tower concordance (for survey and references, we refer [CST11]). It is not our purpose to study this asymmetric filtration theory of Conant-Schneiderman-Teichner. We focus on the finer equivalence relations, symmetric Whitney tower/grope concordance.

Our main result, Theorem A, says that the Friedl-Powell invariant $\tau(L, \chi)$ can be understood in terms of symmetric Whitney tower/grope concordance as conjectured in [FP14, Remark 1.3.(5)]:

Theorem A. *Suppose that L is a 2-component link with linking number 1 and H is the Hopf link. If L and H are height 3.5 Whitney tower (or grope) concordant, then the Friedl-Powell invariant $\tau(L, \chi)$ vanishes for L in the sense of Definition 4.2.*

In the proof, we use the notion of *h -solvable cobordism*, introduced by J. C. Cha in [Cha14] (for the definition, see Section 3.1). By [Cha14, Theorem 2.13], if two links L and L' are height $(h + 2)$ Whitney tower/grope concordant, then their exteriors X_L and $X_{L'}$ are h -solvable cobordant for all $h \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. Actually, we prove Theorem A in Section 4.3 under the (potentially) weaker assumption that there exists a 1.5-solvable cobordism between the exteriors X_L and X_H .

Remark.

- (1) In [COT03, Theorem 9.11], Cochran, Orr, and Teichner proved that if a knot K bounds a Whitney tower/grope of height 3.5 in D^4 , or more generally if K is 1.5-solvable, then the Casson-Gordon invariant $\tau(K, \chi)$ vanishes. Our result can be viewed as an analogue for 2-component links with linking number 1.

- (2) Theorem A is strictly stronger than [FP14, Theorem 3.5] by the following known fact: for any integer $n > 2$, there are links which are height n grope concordant to H but not height $n.5$ Whitney tower concordant to H (in particular, not concordant to H) [Cha14, Theorem 4.1].

Symmetric Whitney tower/grope concordance and abelian invariants

In [COT03, Theorem 1.1], Cochran, Orr, and Teichner proved that a Seifert form of a knot K is metabolic if and only if K bounds a height 2.5 grope in D^4 . By [Sch06, Corollary 2] and [COT03, Theorem 8.12], this condition is equivalent to that K bounds a height 2.5 Whitney tower in D^4 . Motivated from this result, in Section 5, we prove that Blanchfield form and the multivariable Alexander polynomial are actually obstructions to height 3 Whitney tower/grope concordance.

Abelian link concordance invariants are studied by A. Kawauchi [Kaw78] and J. Hillman [Hil12]. To state our main result, we recall their notations (for details, see Section 5) and main results. Let L be a μ -component link and X_L be the exterior of L . Denote $\mathbb{Z}[t_1^{\pm}, \dots, t_{\mu}^{\pm}]$ by Λ_{μ} . The ring Λ_{μ} is endowed with the involution $-: t_i \mapsto t_i^{-1}$. Let S be the multiplicative set generated by $\{t_1 - 1, \dots, t_{\mu} - 1\}$. Denote by $\Lambda_{\mu S}$ the localization of Λ_{μ} with respect to S . Let \mathcal{K} be the quotient field of Λ_{μ} . Using the Hurewicz map $\pi_1(X_L) \rightarrow \mathbb{Z}^{\mu}$, we define $H_*(X_L; \Lambda_{\mu})$ and $H_*(X_L; \Lambda_{\mu S})$.

In [Hil12, Chapter 2], Hillman defined $\mathcal{K}/\Lambda_{\mu S}$ -valued the localized Blanchfield form b_L defined on the quotient of the torsion submodule of $H_1(X_L; \Lambda_{\mu S})$ by its maximal pseudonull-submodule. Also, he proved that the Witt-class of b_L , denoted by $[b_L]$, in the Witt group $W(\mathcal{K}, \Lambda_{\mu S}, -)$ is a link concordance invariant.

In [Kaw78], Kawauchi defined the torsion Alexander polynomial of L which we denote it by Δ_L^T . In [Kaw78, Theorems A, B], he proved that if two links L_0 and L_1 are concordant, then $\text{rank}_{\Lambda_{\mu}} H_1(X_{L_0}; \Lambda_{\mu}) = \text{rank}_{\Lambda_{\mu}} H_1(X_{L_1}; \Lambda_{\mu})$ and $\Delta_{L_0}^T f_0 \overline{f_0} \doteq \Delta_{L_1}^T f_1 \overline{f_1}$ for some $f_i(t_1, \dots, t_{\mu}) \in \Lambda_{\mu}$, $i = 0, 1$ with $|f_i(1, \dots, 1)| = 1$.

We extend these theorems of Hillman and Kawauchi in terms of symmetric Whitney tower/grope concordance as follows:

Theorem B. *Suppose that two links L_0 and L_1 are height 3 Whitney tower/grope concordant. Then, $[b_{L_0}] = [b_{L_1}] \in W(\mathcal{K}, \Lambda_{\mu S}, -)$.*

Theorem C. *Suppose that two links L_0 and L_1 are height 3 Whitney tower/grope concordant. Then,*

- (1) $\text{rank}_{\Lambda_{\mu}} H_1(X_{L_0}; \Lambda_{\mu}) = \text{rank}_{\Lambda_{\mu}} H_1(X_{L_1}; \Lambda_{\mu})$ and
- (2) $\Delta_{L_0}^T f_0 \overline{f_0} \doteq \Delta_{L_1}^T f_1 \overline{f_1}$ for some $f_i(t_1, \dots, t_{\mu}) \in \Lambda_{\mu}$, $i = 0, 1$ with $|f_i(1, \dots, 1)| = 1$.

As a special case of Theorems B and C for 2-component links with linking number 1, we have the following special case. This illustrates that the concordance problem between 2-component link with linking number 1 and the Hopf link is similar with the concordance problem between knot and the unknot.

Corollary D. *Suppose that L is a 2-component link with linking number 1 and H is the Hopf link. If L and H are height 3 Whitney tower/grope concordant, then*

- (1) $[b_L] = 0 \in W(\mathcal{K}, \Lambda_2, -)$,
- (2) $\text{rank}_{\Lambda_2} H_1(X_L; \Lambda_2) = 0$,
- (3) $\Delta_L^T \doteq f \overline{f}$ for some $f(t_1, t_2) \in \Lambda_2$ such that $|f(1, 1)| = 1$.

Remark. Theorems B and C should be compared to the following equivalent statements for knots about abelian invariants. (e.g. [COT03, Theorem 1.1] and [Cha07, Theorem 5.10].)

- (1) The knot K bounds a grope of height 2.5 in D^4 .
- (2) The 0-surgery manifold of K , M_K is 0.5 solvable.
- (3) The Seifert form of K is metabolic (or K is algebraically slice).
- (4) The Blanchfield form of K is Witt-trivial.

Therefore, the most natural assumption for Theorems B and C might be the existence of 0.5-solvable cobordism between link exteriors. The proof for the knot case heavily relies on the existence of Seifert surfaces for K . For general links, as substitutes for Seifert surfaces, there are *immersed* Cooper surfaces studied in [Coo82] (or its generalization in [Cim04]). However, because of their singularities, the similar approach using Cooper surface seems somewhat difficult.

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2. Casson-Gordon type representations

The goal of this section is to prove Theorem 2.3 which will give the key dimension estimate in the proof of Theorem A. Lemma 2.2 and Theorem 2.3 are inspired by [Cha13, Lemma 3.10 and Theorem 3.11]. In the proof of Lemma 2.2, we use the injectivity theorem of Friedl and Powell [FP12, Theorem 3.1] stated in Lemma 2.1.

We recall the notations used in [FP12] for the convenience of the reader. Let $\varphi: G \rightarrow A$ be a surjective group homomorphism where A is a finite abelian p -group. Assume that $\varphi: G \rightarrow A$ factors through a surjective homomorphism $\phi': G \rightarrow \mathcal{H}'$ to a torsion free abelian group \mathcal{H}' . Let $K = \text{Ker } \varphi$, $\mathcal{H} = \text{Im}(\phi'|_K)$ and $\phi: K \rightarrow \mathcal{H}$ be the restriction of ϕ' to K . Note that \mathcal{H} is a torsion free abelian group. In short, we have the following commutative diagram:

$$\begin{array}{ccccccc}
 1 & \longrightarrow & K & \longrightarrow & G & \xrightarrow{\varphi} & A \longrightarrow 1 \\
 & & \downarrow \phi = \phi'|_K & & \downarrow \phi' & \nearrow & \\
 & & \mathcal{H} & \longrightarrow & \mathcal{H}' & &
 \end{array}$$

Suppose that $\alpha: K \rightarrow \text{GL}(d, Q)$ is a d -dimensional representation to a field Q of characteristic zero such that $\alpha|_{\text{Ker } \phi}$ factors through a q -group for some prime q . Let $Q(\mathcal{H})$ be the quotient field of the group ring $Q[\mathcal{H}]$. Note that α and ϕ give the right $\mathbb{Z}K$ -module structure on $Q^d \otimes_Q Q(\mathcal{H}) = Q(\mathcal{H})^d$ as follows:

$$\begin{aligned}
 (*) \quad & Q^d \otimes_Q Q(\mathcal{H}) \times \mathbb{Z}K \longrightarrow Q^d \otimes_Q Q(\mathcal{H}) \\
 & (v \otimes p, g) \longmapsto (v \cdot \alpha(g) \otimes p, \phi(g))
 \end{aligned}$$

We write $t = |A|$. $\mathbb{Z}G$ is a left (rank t free) $\mathbb{Z}K$ -module. Note that there is a right action of G on dt -dimensional Q -vector space $Q^d \otimes_{\mathbb{Z}K} \mathbb{Z}G$. Equivalently, there is an induced representation $\alpha': G \rightarrow \text{GL}(dt, Q)$.

As in (*), α' and ϕ' give the right $\mathbb{Z}G$ -module structure on $Q(\mathcal{H}')^{dt} = Q^{dt} \otimes_Q Q(\mathcal{H}')$:

$$(**) \quad \begin{aligned} Q^{dt} \otimes_Q Q(\mathcal{H}') \times \mathbb{Z}G &\longrightarrow Q^{dt} \otimes_Q Q(\mathcal{H}') \\ (v \otimes p, g) &\longmapsto (v \cdot \alpha'(g) \otimes p, \phi'(g)) \end{aligned}$$

Regard \mathbb{Z}/q as a $\mathbb{Z}G$ -module with the trivial G -action. Now, we state [FP12, Theorem 3.1].

Lemma 2.1. [FP12, Theorem 3.1] *Let $f: M \rightarrow N$ be a morphism of projective left $\mathbb{Z}G$ -modules such that*

$$1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} f: \mathbb{Z}/q \otimes_{\mathbb{Z}G} M \longrightarrow \mathbb{Z}/q \otimes_{\mathbb{Z}G} N$$

is injective. Then,

$$1_{Q(\mathcal{H}')^{dt}} \otimes_{\mathbb{Z}G} f: Q(\mathcal{H}')^{dt} \otimes_{\mathbb{Z}G} M \longrightarrow Q(\mathcal{H}')^{dt} \otimes_{\mathbb{Z}G} N$$

is injective.

Using Lemma 2.1, we prove Lemma 2.2 and Theorem 2.3.

Lemma 2.2. *Let $f: M \rightarrow N$ be a morphism of left $\mathbb{Z}G$ -modules.*

(1) *If N is projective, then*

$$dt \cdot \dim_{\mathbb{Z}/q} \text{Im}(1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} f) \leq \dim_{Q(\mathcal{H}')} \text{Im}(1_{Q(\mathcal{H}')^{dt}} \otimes_{\mathbb{Z}G} f).$$

(2) *If, in addition, M is finitely generated and free, then*

$$dt \cdot \dim_{\mathbb{Z}/q} \text{Ker}(1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} f) \geq \dim_{Q(\mathcal{H}')} \text{Ker}(1_{Q(\mathcal{H}')^{dt}} \otimes_{\mathbb{Z}G} f).$$

Proof. (1) Let $k = \dim_{\mathbb{Z}/q} \text{Im}(1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} f)$. (k may be any cardinal number.) Note that $\mathbb{Z}/q \otimes_{\mathbb{Z}G} -$ induces two surjections $(\mathbb{Z}G)^k \rightarrow \mathbb{Z}/q^k$ and $\text{Im } f \rightarrow \text{Im}(1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} f)$. Since $(\mathbb{Z}G)^k$ is free, the following diagram commutes:

$$\begin{array}{ccc} (\mathbb{Z}G)^k & \xrightarrow{\exists \ i} & \text{Im } f \\ \downarrow & & \downarrow \\ \mathbb{Z}/q^k & \xrightarrow{\cong} & \text{Im}(1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} f) \end{array}$$

Recall that N is a projective $\mathbb{Z}G$ -module. Obviously, $(\mathbb{Z}G)^k$ is a projective $\mathbb{Z}G$ -module. Hence, we can apply Lemma 2.1 to $i: (\mathbb{Z}G)^k \rightarrow \text{Im } f \subset N$ and obtain the following injection:

$$1_{Q(\mathcal{H}')^{dt}} \otimes_{\mathbb{Z}G} i: Q(\mathcal{H}')^{dtk} = Q(\mathcal{H}')^{dt} \otimes_{\mathbb{Z}G} (\mathbb{Z}G)^k \hookrightarrow Q(\mathcal{H}')^{dt} \otimes_{\mathbb{Z}G} N$$

Since $\text{Im } i \subset \text{Im } f$,

$$Q(\mathcal{H}')^{dtk} \cong \text{Im}(1_{Q(\mathcal{H}')^{dt}} \otimes_{\mathbb{Z}G} i) \subset \text{Im}(1_{Q(\mathcal{H}')^{dt}} \otimes_{\mathbb{Z}G} f).$$

This implies

$$dt \cdot \dim_{\mathbb{Z}/q} \text{Im}(1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} f) = dtk \leq \dim_{Q(\mathcal{H}')} (1_{Q(\mathcal{H}')^{dt}} \otimes_{\mathbb{Z}G} f).$$

(2) Let $M = (\mathbb{Z}G)^n$. (1) and the following elementary observation completes the proof.

$$\dim_{\mathbb{Z}/q} \text{Ker}(1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} f) + \dim_{\mathbb{Z}/q} \text{Im}(1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} f) = \dim_{\mathbb{Z}/q} \mathbb{Z}/q \otimes_{\mathbb{Z}G} M = n$$

and similarly,

$$ndt = \dim_{Q(\mathcal{H}')} \text{Ker}(1_{Q(\mathcal{H}')^{dt}} \otimes_{\mathbb{Z}G} f) + \dim_{Q(\mathcal{H}')} \text{Im}(1_{Q(\mathcal{H}')^{dt}} \otimes_{\mathbb{Z}G} f).$$

□

Theorem 2.3. *Suppose C_* is a chain complex of projective left $\mathbb{Z}G$ -modules with C_n finitely generated. If $\{x_i\}_{i \in I}$ is a collection of n -cycles in C_n , then for the $(Q(\mathcal{H})^d \otimes_{\mathbb{Z}K} \mathbb{Z}G)$ -span of $\{[1_{Q(\mathcal{H})^d} \otimes_{\mathbb{Z}K} x_i]\}_{i \in I}$, $M \subset H_n(Q(\mathcal{H})^d \otimes_{\mathbb{Z}K} C_*)$ and the \mathbb{Z}/q -span of $\{[1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} x_i]\}_{i \in I}$, $\overline{M} \subset H_n(\mathbb{Z}/q \otimes_{\mathbb{Z}G} C_*)$, we have*

$$\dim_{Q(\mathcal{H})} H_n(Q(\mathcal{H})^d \otimes_{\mathbb{Z}K} C_*)/M \leq dt \cdot \dim_{\mathbb{Z}/q} H_n(\mathbb{Z}/q \otimes_{\mathbb{Z}G} C_*)/\overline{M}.$$

Proof. Let $\partial_n: C_n \rightarrow C_{n-1}$ be the boundary map of C_* and define $f: (\mathbb{Z}G)^I \oplus C_{n+1} \rightarrow C_n$ by $(e_i, v) \mapsto x_i + \partial_{n+1}(v)$, where $\{e_i\}_{i \in I}$ is the standard basis of $(\mathbb{Z}G)^I$. Then,

$$\begin{aligned} H_n(Q(\mathcal{H})^d \otimes_{\mathbb{Z}K} C_*)/M &= \text{Ker}(1_{Q(\mathcal{H})^d} \otimes_{\mathbb{Z}K} \partial_n) / \text{Im}(1_{Q(\mathcal{H})^d} \otimes_{\mathbb{Z}K} f) \text{ and} \\ H_n(\mathbb{Z}/q \otimes_{\mathbb{Z}G} C_*)/\overline{M} &= \text{Ker}(1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} \partial_n) / \text{Im}(1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} f). \end{aligned}$$

From the $\mathbb{Z}G$ -module structure on $Q(\mathcal{H}')^{dt}$ in (**),

$$Q(\mathcal{H}') \otimes_{Q(\mathcal{H})} Q(\mathcal{H})^d \otimes_{\mathbb{Z}K} \mathbb{Z}G = (Q(\mathcal{H}') \otimes_{Q(\mathcal{H})} Q(\mathcal{H})^d \otimes_{\mathbb{Z}K} \mathbb{Z}G) \otimes_{\mathbb{Z}G} \mathbb{Z}G = Q(\mathcal{H}')^{dt} \otimes_{\mathbb{Z}G} \mathbb{Z}G.$$

Since C_* is a chain complex of left $\mathbb{Z}G$ -modules,

$$Q(\mathcal{H}') \otimes_{Q(\mathcal{H})} Q(\mathcal{H})^d \otimes_{\mathbb{Z}K} C_* = Q(\mathcal{H}')^{dt} \otimes_{\mathbb{Z}G} C_*.$$

Since $\mathcal{H} \hookrightarrow \mathcal{H}'$, $Q(\mathcal{H}')$ is a flat right $Q(\mathcal{H})$ -module. Therefore, we have

$$H_*(Q(\mathcal{H}')^{dt} \otimes_{\mathbb{Z}G} C_*) = Q(\mathcal{H}') \otimes_{Q(\mathcal{H})} H_*(Q(\mathcal{H})^d \otimes_{\mathbb{Z}K} C_*).$$

Combining these, we obtain

$$\dim_{Q(\mathcal{H})} H_n(Q(\mathcal{H})^d \otimes_{\mathbb{Z}K} C_*)/M = \dim_{Q(\mathcal{H}')} H_n(Q(\mathcal{H}')^{dt} \otimes_{\mathbb{Z}G} C_*) / (Q(\mathcal{H}') \otimes_{Q(\mathcal{H})} M)$$

and

$$H_n(Q(\mathcal{H}')^{dt} \otimes_{\mathbb{Z}G} C_*) / (Q(\mathcal{H}') \otimes_{Q(\mathcal{H})} M) = \text{Ker}(1_{Q(\mathcal{H}')^{dt}} \otimes_{\mathbb{Z}G} \partial_n) / \text{Im}(1_{Q(\mathcal{H}')^{dt}} \otimes_{\mathbb{Z}G} f).$$

From the above observations and the inequality from Lemma 2.2,

$$\begin{aligned} & \dim_{Q(\mathcal{H})} H_n(Q(\mathcal{H})^d \otimes_{\mathbb{Z}K} C_*)/M \\ &= \dim_{Q(\mathcal{H}')} H_n(Q(\mathcal{H}')^{dt} \otimes_{\mathbb{Z}G} C_*) / (Q(\mathcal{H}') \otimes_{Q(\mathcal{H})} M) \\ &= \dim_{Q(\mathcal{H}')} \text{Ker}(1_{Q(\mathcal{H}')^{dt}} \otimes_{\mathbb{Z}G} \partial_n) - \dim_{Q(\mathcal{H}')} \text{Im}(1_{Q(\mathcal{H}')^{dt}} \otimes_{\mathbb{Z}G} f) \\ &\leq dt(\dim_{\mathbb{Z}/q} \text{Ker}(1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} \partial_n) - \dim_{\mathbb{Z}/q} \text{Im}(1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} f)) \\ &= dt \cdot \dim_{\mathbb{Z}/q} H_n(\mathbb{Z}/q \otimes_{\mathbb{Z}G} C_*)/\overline{M}. \end{aligned}$$

This completes the proof. □

3. h -solvable cobordism

In this section, we give the definition of an h -solvable cobordism following [Cha14]. Also, we prove Proposition 3.2 about prime power covering of 1-solvable cobordism.

3.1. Definition of h -solvable cobordism

For oriented compact bordered 3-manifolds M and M' with a chosen homeomorphism $\partial M \cong \partial M'$, a *cobordism* W between M and M' is a 4-dimensional manifold with boundary $\partial W = M \cup_{\partial} -M'$, where $-M'$ denotes M' with reversed orientation. We often denote a cobordism by $(W; M, M')$. A cobordism $(W; M, M')$ is an H_1 -cobordism (resp. a *homology cobordism*) if $H_i(M; \mathbb{Z}) \cong H_i(W; \mathbb{Z}) \cong H_i(M'; \mathbb{Z})$ under the inclusion map for $i \leq 1$ (resp. for all i). Note that $H_2(W, M)$ is a free abelian group if $(W; M, M')$ is an H_1 -cobordism (for example, see [Cha13, Lemma 3.7]).

Example. If L is a link in S^3 , then the link exterior X_L is a bordered 3-manifold with a canonical homeomorphism between disjoint union of tori and ∂X_L sending standard basis to the meridians and 0-framed longitudes of L . If two links L and L' are concordant, then X_L and $X_{L'}$ are homology cobordant bordered 3-manifolds via a concordance exterior and h -solvable cobordant for all $h \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ (see the definition of solvable cobordism given in below).

We use the following notation for covering maps associated to the derived series.

Convention.

- (1) For a space X , there is a sequence of regular covers over X

$$X^{(n+1)} \longrightarrow X^{(n)} \longrightarrow \cdots \longrightarrow X^{(1)} \longrightarrow X^{(0)} = X$$

which corresponds to the derived series

$$\pi^{(n+1)} \subset \pi^{(n)} \subset \cdots \subset \pi^{(0)} = \pi, \text{ where } \pi = \pi_1(X) \text{ and } \pi^{(n+1)} = [\pi^{(n)}, \pi^{(n)}].$$

With this, we can always identify $H_*(X; \mathbb{Z}[\pi/\pi^{(n)}]) = H_*(X^{(n)}; \mathbb{Z})$ as usual.

- (2) For a 4-manifold W with $\pi = \pi_1(W)$, let

$$\lambda_n: H_2(W; \mathbb{Z}[\pi/\pi^{(n)}]) \times H_2(W; \mathbb{Z}[\pi/\pi^{(n)}]) \longrightarrow \mathbb{Z}[\pi/\pi^{(n)}]$$

be the inetersection form.

- (3) For a covering map $Y \rightarrow X$, $\text{Cov}(Y|X)$ denotes its deck transformation group. Assume that the action of $\text{Cov}(Y|X)$ on $H_*(Y; \mathbb{Z})$ is a right action.

Definition 3.1. Suppose $(W; M, M')$ is an H_1 -cobordism between bordered 3-manifolds M and M' with $\pi = \pi_1(W)$. Let $r = \frac{1}{2} \text{rank } H_2(W, M; \mathbb{Z})$.

- (1) A submodule $L \subset H_2(W; \mathbb{Z}[\pi/\pi^{(n)}])$ is an n -lagrangian if L projects to a half-rank summand of $H_2(W, M; \mathbb{Z})$ and λ_n vanishes on L .
- (2) For an n -lagrangian L ($k \leq n$), homology classes $d_1, \dots, d_r \in H_2(W; \mathbb{Z}[\pi/\pi^{(k)}])$ are k -duals if L is generated by $l_1, \dots, l_r \in L$ whose projections $l'_1, \dots, l'_r \in H_2(W; \mathbb{Z}[\pi/\pi^{(k)}])$ satisfy $\lambda_k(l'_i, d_j) = \delta_{ij}$.
- (3) An H_1 -cobordism $(W; M, M')$ is called an $n.5$ -solvable cobordism (resp. n -solvable cobordism) if it has an $(n+1)$ -lagrangian (resp. n -lagrangian) admitting n -duals. If there exists an h -solvable cobordism from M to M' , we say that M is h -solvable cobordant to M' for $h \in \frac{1}{2}\mathbb{Z}_{\geq 0}$.

3.2. Prime power cover of 1-solvable cobordism

In this subsection, we prove Proposition 3.2 about (abelian) prime power cover of 1-solvable cobordism for later purpose.

Proposition 3.2. Suppose that $(W; M, M')$ is a 1-solvable cobordism with $\varphi: \pi_1 W \rightarrow A$ be a surjective group homomorphism to an abelian p -group A and p is prime. We denote the cobordism of the induced coverings by $(W^\varphi; M^\varphi, M'^\varphi)$. Then,

- (1) $\beta_2(W^\varphi, M^\varphi) = |A|\beta_2(W, M)$ where β_2 is the second Betti number.
- (2) The inclusion induced map $FH_2(W^\varphi; \mathbb{Z}) \rightarrow FH_2(W^\varphi, M^\varphi; \mathbb{Z})$ is surjective.
- (3) $(W^\varphi; M^\varphi, M'^\varphi)$ is an H_1 -cobordism with \mathbb{Q} -coefficients.

Here, for a finitely generate abelian group G , FG denotes the free part of G .

Proof. (1) Fix a (relative) CW-complex structure of (W, M) . This induces a (relative) CW-complex structure of (W^φ, M^φ) . Let $C_* = C_*(W^\varphi, M^\varphi; \mathbb{Z})$. Then, C_* is a chain complex of free $\mathbb{Z}A$ -modules and $C_*(W, M; \mathbb{Z}) = C_* \otimes_{\mathbb{Z}A} \mathbb{Z}$. Since $(W; M, M')$ is an H_1 -cobordism, $H_i(C_* \otimes_{\mathbb{Z}A} \mathbb{Z}/p) = 0$ for $i = 0, 1$ by universal coefficient theorem. Since p is prime, the well-known Levine's chain homotopy lifting argument in [Lev94] shows that $H_i(C_* \otimes_{\mathbb{Z}} \mathbb{Z}/p) = 0$ for $i = 0, 1$. In particular, by universal coefficient theorem, $H_i(C_*)$ is a torsion abelian group for $i = 0, 1$. By universal coefficient theorem, $H_i(W^\varphi, M^\varphi; \mathbb{Q}) = H_i(C_*) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ for $i = 0, 1$.

By taking $C_* = C_*(W^\varphi, M'^\varphi; \mathbb{Z})$, the same argument shows $H_i(W^\varphi, M'^\varphi; \mathbb{Q}) = 0$ for $i = 0, 1$. By Poincaré duality and universal coefficient theorem,

$$H_i(W^\varphi, M^\varphi; \mathbb{Q}) \cong \text{Hom}_{\mathbb{Q}}(H_{4-i}(W^\varphi, M'^\varphi; \mathbb{Q}), \mathbb{Q}) = 0 \text{ for } i = 3, 4.$$

So, $\beta_2(W^\varphi, M^\varphi) = \chi(W^\varphi, M^\varphi)$ where χ is the Euler characteristic. Similarly, $\chi(W, M) = \beta_2(W, M)$ because $H_i(W, M; \mathbb{Z}) = H_i(W, M'; \mathbb{Z}) = 0$ for $i = 0, 1$. By definition, (W^φ, M^φ) is an A -cover of (W, M) and $\chi(W^\varphi, M^\varphi) = |A|\chi(W, M)$. This completes the proof of (1).

(2) Since $W^\varphi \rightarrow W$ is an abelian covering with $\text{Cov}(W^\varphi|W) = A$, $\pi_1(W)^{(1)} \subset \pi_1(W^\varphi)$. The covering map $W^{(1)} \rightarrow W^\varphi$ induces $H_2(W^{(1)}; \mathbb{Z}) \rightarrow H_2(W^\varphi; \mathbb{Z})$. Let $l_1, \dots, l_r, d_1, \dots, d_r$ be the images of the (generators of) 1-lagrangian and 1-duals in $H_2(W^\varphi; \mathbb{Z})$. By the definition of 1-solvable cobordism, $\beta_2(W, M) = 2r$. Let $A = \{g_1, \dots, g_t\}$. From (1), $\beta_2(W^\varphi, M^\varphi) = \beta_2(W, M)|A| = 2rt$.

From the (right) group action of A on $H_2(W^\varphi; \mathbb{Z})$, we can define

$$l_{ij} = l_i \cdot g_j \text{ and } d_{kl} = d_k \cdot g_l \text{ for } 1 \leq i, k \leq r \text{ and } 1 \leq j, l \leq t.$$

By the definition of 1-Lagrangian and 1-duals, the intersection pairing $\lambda: FH_2(W^\varphi, M^\varphi; \mathbb{Z}) \times FH_2(W^\varphi, M'^\varphi; \mathbb{Z}) \rightarrow \mathbb{Z}$ restricted to the \mathbb{Z} -span of (the image of) $\{l_{ij}, d_{kl}\}$ is

$$\begin{pmatrix} 0 & I_{rt \times rt} \\ I_{rt \times rt} & X \end{pmatrix}$$

By rank counting, the image of $\{l_{ij}, d_{kl}\}$ form a \mathbb{Z} -basis of $FH_2(W^\varphi, M^\varphi; \mathbb{Z})$. This proves that $\text{inc}_*: FH_2(W^\varphi; \mathbb{Z}) \rightarrow FH_2(W^\varphi, M^\varphi; \mathbb{Z})$ is surjective because $\{l_{ij}, d_{kl}\} \subset FH_2(W^\varphi; \mathbb{Z})$.

(3) From (2), $\text{inc}_*: H_2(W^\varphi; \mathbb{Q}) \rightarrow H_2(W^\varphi, M^\varphi; \mathbb{Q})$ is surjective. From (1) and the homology long exact sequence of a pair (W^φ, M^φ) , $\text{inc}_*: H_i(M^\varphi; \mathbb{Q}) \rightarrow H_i(W^\varphi; \mathbb{Q})$ is an isomorphism for $i = 0, 1$. Same argument works for (W, M') . This completes the proof. \square

4. Solvable cobordism and Friedl-Powell invariant

Throughout this section, for any finitely generated abelian group G , tG and FG denote the torsion part of G and free part of G , respectively. G^\wedge denotes $\text{Hom}_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z})$. For a finite abelian group G , $G^\wedge = \text{Ext}_{\mathbb{Z}}(G, \mathbb{Z})$ since $\text{Hom}_{\mathbb{Z}}(G, \mathbb{Q}) = \text{Ext}_{\mathbb{Z}}(G, \mathbb{Q}) = 0$. $H_*(-)$ denotes homology with integral coefficients.

4.1. Definition of the Friedl-Powell invariant $\tau(L, \chi)$

To define the Friedl-Powell invariant $\tau(L, \chi)$, we set up notations and conventions used in [FP14]. Here, L is a 2-component link with linking number 1 and H is the Hopf link. We

denote the exterior of J by $X_J = S^3 - \nu(J)$ for $J = H, L$. We can decompose ∂X_L into $Y_a \cup Y_b$ with $Y_a \cong Y_b = S^1 \times D^1 \sqcup S^1 \times D^1$ where both Y_a and Y_b are annuli neighborhood of (parallels of) meridians of L . Define $M_J = X_J \cup_{\partial X_H \times I} -X_H$ for $J = H, L$ where the gluing map respects the ordering of the link components and identifies each of the subsets $Y_a, Y_b \subset \partial X_J$ for $J = L, H$.

For a prime p , we say a group homomorphism $\varphi: H_1(M_L) \rightarrow \mathbb{Z}/p^i \oplus \mathbb{Z}/p^j$ is *admissible* if φ sends two meridians of L to the standard basis $(1, 0), (0, 1)$. (From the Mayer-Vietoris sequence, $H_1(M_L) \cong H_1(X_L) \oplus \mathbb{Z} \cong \mathbb{Z}^3$.) Let $M_L^\varphi \rightarrow M_L$ be the p^{i+j} -fold covering space from φ . We denote the Hurewicz map by $\phi': \pi_1(M_L) \rightarrow H_1(M_L)$. Define $\phi: \pi_1(M_L^\varphi) \rightarrow H_1(M_L)$ be the restriction $\phi'|_{\pi_1(M_L^\varphi)}$ and $\mathcal{H} = \text{Im } \phi$. Choose an isomorphism $\psi: \pi_1(T^3) \cong \mathcal{H}$. (Note that \mathcal{H} is isomorphic to \mathbb{Z}^3 as a finite index subgroup of $H_1(M_L) \cong \mathbb{Z}^3$.)

For a prime power character $\chi: \pi_1(M_L^\varphi) \rightarrow \mathbb{Z}/q^k$, we have the bordism class

$$[(M_L^\varphi, \chi \times \phi) \sqcup -(T^3, \text{tr} \times \psi)] \in \Omega_3(\mathbb{Z}/q^k \times \mathcal{H})$$

where $\text{tr}: \pi_1(T^3) \rightarrow \mathbb{Z}/q^k$ is the trivial group homomorphism. From the Atiyah-Hirzebruch spectral sequence calculation given in [FP14, Section 3.2], $[(M_L^\varphi, \chi \times \phi) \sqcup -(T^3, \text{tr} \times \psi)]$ is q -primary torsion in $\Omega_3(\mathbb{Z}/q^k \times \mathcal{H})$. In other words, there exist a non-negative integer s , a cobordism W between $q^s M_L^\varphi$ and $q^s T^3$, and $\Phi: \pi_1(W) \rightarrow \mathbb{Z}/q^k \times \mathcal{H}$ such that the following diagram commutes:

$$\begin{array}{ccc} \bigsqcup^{q^s} (M_L^\varphi \sqcup -T^3) & \xrightarrow{\chi \times \phi \sqcup \text{tr} \times \psi} & K(\mathbb{Z}/q^k \times \mathcal{H}, 1) \\ \downarrow \partial & \nearrow \Phi & \\ W & & \end{array}$$

From the following sequence of ring homomorphisms,

$$\mathbb{Z}[\pi_1(W)] \xrightarrow{\Phi} \mathbb{Z}[\mathbb{Z}/q^k \times \mathcal{H}] = \mathbb{Z}[\mathbb{Z}/q^k][\mathcal{H}] \longrightarrow \mathbb{Q}(\xi_{q^k})(\mathcal{H}) \longrightarrow \mathbb{C}(\mathcal{H}) = \mathcal{K}$$

we can define the twisted intersection form $H_2(W; \mathcal{K}) \times H_2(W; \mathcal{K}) \rightarrow \mathcal{K}$. We denote the non-singular part of the intersection form on $H_2(W; \mathcal{K})$ (resp. on $H_2(W; \mathbb{Q})$) by $\lambda_{\mathcal{K}}(W)$ (resp. $\lambda_{\mathbb{Q}}(W)$).

Definition 4.1 (Friedl-Powell invariant).

$$\tau(L, \chi) = (\lambda_{\mathcal{K}}(W) - \mathcal{K} \otimes \lambda_{\mathbb{Q}}(W)) \otimes \frac{1}{q^s} \in L^0(\mathcal{K}) \otimes_{\mathbb{Z}} \mathbb{Z}[1/q]$$

where $L^0(\mathcal{K})$ is the Witt group of finite dimensional non-singular sesquilinear forms over \mathcal{K} .

In [FP14, Section 3.2], it is shown that $\tau(L, \chi)$ is well-defined. That is, $\tau(L, \chi)$ depend neither on the choice of W nor on the choice of isomorphism $\psi: \pi_1(T^3) \rightarrow \mathcal{H}$.

In Section 4.2, we will describe the linking form

$$\lambda_L: tH_1(X_L^\varphi, Y_a^\varphi) \times tH_1(X_L^\varphi, Y_a^\varphi) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

Now, we give the precise statement that *the Friedl-Powell invariant vanishes*.

Definition 4.2. For 2-component link L with linking number 1, we say *the Friedl-Powell invariant vanishes for L* if for any admissible homomorphism $\varphi: H_1(M_L) \rightarrow \mathbb{Z}/p^i \oplus \mathbb{Z}/p^j$ and for a prime p , there exists a metabolizer $P = P^\perp$ of the linking form

$$\lambda_L: tH_1(X_L^\varphi, Y_a^\varphi) \times tH_1(X_L^\varphi, Y_a^\varphi) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

with the following property: for any character of prime power order $\chi: H_1(M_L^\varphi) \rightarrow \mathbb{Z}/q^k$ which satisfies that $\chi|_{H_1(X_L^\varphi)}$ factors through

$$\begin{array}{ccc} H_1(X_L^\varphi) & \xrightarrow{\chi|_{H_1(X_L^\varphi)}} & \mathbb{Z}/q^k \\ \downarrow & \nearrow \delta & \\ H_1(X_L^\varphi, Y_a^\varphi) & & \end{array}$$

and that δ vanishes on P , $\tau(L; \chi) = 0 \in L^0(\mathcal{K}) \otimes_{\mathbb{Z}} \mathbb{Z}[1/q]$.

The following main theorem will be proved in Section 4.3:

Theorem A. *Suppose that L is a 2-component link with linking number 1 and H is the Hopf link. If X_L and X_H are 1.5-solvable cobordant, then the Friedl-Powell invariant $\tau(L, \chi)$ vanishes for L in the sense of Definition 4.2. In particular, the conclusion holds if L and H are height 3.5 Whitney tower/grope concordant.*

4.2. 1-solvable cobordism and a metabolizer of the linking form

In this subsection, we recall the definition of the linking form

$$\lambda_L: tH_1(X_L^\varphi, Y_a^\varphi) \times tH_1(X_L^\varphi, Y_a^\varphi) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

defined in [FP14] and prove Proposition 4.3. The adjoint of λ_L , $\text{Ad}(\lambda_L): tH_1(X_L^\varphi, Y_a^\varphi) \rightarrow tH_1(X_L^\varphi, Y_a^\varphi)^\wedge$, can be obtained by composing the following isomorphisms:

$$(a) \quad tH_1(X_L^\varphi, Y_a^\varphi) \longrightarrow tH^2(X_L^\varphi, Y_b^\varphi) \longrightarrow \text{Ext}_{\mathbb{Z}}(tH_1(X_L^\varphi, Y_b^\varphi), \mathbb{Z}) = tH_1(X_L^\varphi, Y_a^\varphi)^\wedge.$$

We used Poincaré duality, universal coefficient theorem, $H_1(X_L^\varphi, Y_b^\varphi) \cong H_1(X_L^\varphi, Y_a^\varphi)$, and $tH_1(X_L^\varphi, Y_a^\varphi)$ is a finite abelian group.

Let $(W_0; X_L, X_H)$ be a 1-solvable cobordism. Recall $\varphi: H_1(M_L) \rightarrow \mathbb{Z}/p^i \oplus \mathbb{Z}/p^j$ is an admissible homomorphism and $H_1(M_L) \cong H_1(X_L) \oplus \mathbb{Z}$. Then, $\varphi|_{H_1(X_L)}$ extends to $H_1(W_0)$ canonically because $H_1(X_L) \cong H_1(W_0)$. In this sense, an admissible homomorphism φ always induces a covering $(W_0^\varphi; X_L^\varphi, X_H^\varphi) \rightarrow (W; X_L, X_H)$.

Proposition 4.3. *Suppose $(W_0; X_L, X_H)$ is a 1-solvable cobordism. Let $(W_0^\varphi; X_L^\varphi, X_H^\varphi)$ be a covering induced from an admissible homomorphism $\varphi: H_1(M_L) \rightarrow \mathbb{Z}/p^i \oplus \mathbb{Z}/p^j$, then*

$$P = \text{Ker}(tH_1(X_L^\varphi, Y_a^\varphi) \longrightarrow tH_1(W_0^\varphi, Y_a^\varphi))$$

is a metabolizer of the linking form λ_L .

Proof. Suppose that we have the diagram (b) with two exact rows. Then,

$$P^\perp = (\text{Ad}(\lambda_L))^{-1}(\text{Ker } \partial^\wedge) = \text{Ker}(\text{inc}_*) = P.$$

Hence, it suffices to prove the existence of diagram (b) with two exact rows.

$$(b) \quad \begin{array}{ccccc} tH_2(W_0^\varphi, X_L^\varphi) & \xrightarrow{\partial} & tH_1(X_L^\varphi, Y_a^\varphi) & \xrightarrow{\text{inc}_*} & tH_1(W_0^\varphi, Y_a^\varphi) \\ \theta_1 \downarrow \cong & & \text{Ad}(\lambda_L) \downarrow \cong & & \theta_2 \downarrow \cong \\ tH_1(W_0^\varphi, Y_a^\varphi)^\wedge & \xrightarrow{\text{inc}_*^\wedge} & tH_1(X_L^\varphi, Y_a^\varphi)^\wedge & \xrightarrow{\partial^\wedge} & tH_2(W_0^\varphi, X_L^\varphi)^\wedge \end{array}$$

As in (a), $tH_2(W_0^\varphi, X_L^\varphi) \cong tH_1(W_0^\varphi, X_H^\varphi)^\wedge$ and $tH_1(W_0^\varphi, Y_a^\varphi) \cong tH_2(W_0^\varphi, \partial W_0^\varphi - Y_a^\varphi)^\wedge$ by Poncaré duality and universal coefficient theorem. (Note that $\partial W_0^\varphi = X_L^\varphi \cup X_H^\varphi$.) So, from the following claim, we can define isomorphisms θ_1 and θ_2 .

Claim. *The inclusion maps induce two isomorphisms*

- (1) $tH_1(W_0^\varphi, Y_a^\varphi) \cong tH_1(W_0^\varphi, X_H^\varphi)$ and
- (2) $tH_2(W_0^\varphi, X_L^\varphi) \cong tH_2(W_0^\varphi, \partial W_0^\varphi - Y_a^\varphi)$.

Proof of Claim. By Proposition 3.2 (3), $(W_0^\varphi; X_L^\varphi, X_H^\varphi)$ is an H_1 -cobordism with \mathbb{Q} -coefficients. From this and the proofs of [FP14, Lemmas 2.6, 2.7 and 2.9] (W_0 plays the role of E_C),

$$\text{Coker}(\text{inc}_*: H_1(X_H^\varphi, Y_a^\varphi) \longrightarrow H_1(W_0^\varphi, Y_a^\varphi)) \cong tH_1(W_0^\varphi, Y_a^\varphi).$$

From the homology long exact sequence of a triple $(W_0^\varphi, X_H^\varphi, Y_a^\varphi)$, we have an exact sequence

$$0 \longrightarrow tH_1(W_0^\varphi, Y_a^\varphi) \longrightarrow H_1(W_0^\varphi, X_H^\varphi) \longrightarrow H_0(X_H^\varphi, Y_a^\varphi) = 0$$

which proves (1).

From the proof of [FP14, Lemma 2.5], $tH_1(\partial W_0^\varphi - Y_a^\varphi, X_L^\varphi) = 0$ and the inclusion map $(\partial W_0^\varphi - Y_a^\varphi, X_L^\varphi) \rightarrow (\partial W_0^\varphi, X_L^\varphi)$ induces the zero map on H_2 . In particular,

$$\text{inc}_*: H_2(\partial W_0^\varphi - Y_a^\varphi, X_L^\varphi) \longrightarrow H_2(W_0^\varphi, X_L^\varphi)$$

is also the zero map. From the homology long exact sequence of a triple $(W_0^\varphi, \partial W_0^\varphi - Y_a^\varphi, X_L^\varphi)$,

$$H_2(W_0^\varphi, X_L^\varphi) \cong \text{Ker}(\partial: H_2(W_0^\varphi, \partial W_0^\varphi - Y_a^\varphi) \longrightarrow H_1(\partial W_0^\varphi - Y_a^\varphi, X_L^\varphi)).$$

By taking torsion subgroups, we obtain (2) via

$$\begin{aligned} tH_2(W_0^\varphi, X_L^\varphi) &\cong \text{Ker}(tH_2(W_0^\varphi, \partial W_0^\varphi - Y_a^\varphi) \longrightarrow tH_1(\partial W_0^\varphi - Y_a^\varphi, X_L^\varphi) = 0) \\ &= tH_2(W_0^\varphi, \partial W_0^\varphi - Y_a^\varphi). \end{aligned}$$

□

Commutativity of the diagram (b) also easily follows. For exactness of the first row of (b), we prove the following Lemma.

Lemma 4.4. *Suppose $(W_0; X_L, X_H)$ is a 1-solvable cobordism. We have the following exact sequence*

$$tH_2(W_0^\varphi, X_L^\varphi) \xrightarrow{\partial} tH_1(X_L^\varphi, Y_a^\varphi) \xrightarrow{\text{inc}_*} tH_1(W_0^\varphi, Y_a^\varphi)$$

which is the restriction of a long exact sequence of triple $(W_0^\varphi, X_L^\varphi, Y_a^\varphi)$ to their torsion subgroups.

Proof of the Lemma 4.4. Since $\text{inc}_* \circ \partial = 0$, we prove that $\text{Ker}(\text{inc}_*) \subset \text{Im } \partial$. Let $x \in \text{Ker}(\text{inc}_*)$. By the homology long exact sequence of triple $(W_0^\varphi, X_L^\varphi, Y_a^\varphi)$, there exists $y \in H_2(W_0^\varphi, X_L^\varphi)$ such that $\partial y = x$. By Proposition 3.2 (2), $FH_2(W_0^\varphi) \rightarrow FH_2(W_0^\varphi, X_L^\varphi)$ is surjective. So, $j: FH_2(W_0^\varphi, Y_a^\varphi) \rightarrow FH_2(W_0^\varphi, X_L^\varphi)$ is also surjective. We can choose $z \in FH_2(W_0^\varphi, Y_a^\varphi)$ such that $y - j(z) \in tH_2(W_0^\varphi, X_L^\varphi)$. Then, $\partial(y - j(z)) = \partial y = x$ and this shows that $\text{Ker}(\text{inc}_*) \subset \text{Im } \partial$. □

Note that if $A \xrightarrow{f} B \xrightarrow{g} C$ is an exact sequence of abelian groups. Since \mathbb{Q}/\mathbb{Z} is a divisible group, \mathbb{Q}/\mathbb{Z} is an injective \mathbb{Z} -module. For any abelian group G , $\text{Ext}_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z}) = 0$. Hence, $\text{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ is an exact functor and we obtain $C^\wedge \rightarrow B^\wedge \rightarrow A^\wedge$ is exact. This proves that the second row of the diagram (b) is also exact and completes the proof of Proposition 4.3. □

4.3. Proof of Theorem A

In this subsection, we prove Theorem A. Let $(W_0; X_L, X_H)$ be a 1.5-solvable cobordism with $\beta_2(W_0, X_L) = 2r$. Note that $\partial W_0 = X_L \cup \partial X_H \times I \cup -X_H = M_L$. Attach $X_H \times I$ to W_0 along $\partial X_H \times I$ to get

$$W = W_0 \cup_{\partial X_H \times I} X_H \times I$$

with $\partial W = M_L \sqcup -M_H$. Recall $\varphi: H_1(M_L) \rightarrow \mathbb{Z}/p^i \oplus \mathbb{Z}/p^j$. Applying Mayer-Vietoris argument to $W = W_0 \cup X_H \times I$, the inclusion induces $H_1(M_L) \cong H_1(W)$. So, φ extends to $H_1(W)$ naturally and denote the induced cobordism of coverings by $(W^\varphi; M_L^\varphi, M_H^\varphi)$.

From Proposition 4.3, we can take a metabolizer

$$P := \text{Ker}(tH_1(X_L^\varphi, Y_a^\varphi) \longrightarrow tH_1(W_0^\varphi, Y_a^\varphi))$$

of the linking form λ_L . We fix a character $\chi: H_1(M_L^\varphi) \rightarrow \mathbb{Z}/q^k$ satisfies that $\chi|_{H_1(X_L^\varphi)}$ factors through

$$\begin{array}{ccc} H_1(X_L^\varphi) & \xrightarrow{\chi|_{H_1(X_L^\varphi)}} & \mathbb{Z}/q^k \\ \downarrow & \nearrow \delta & \\ H_1(X_L^\varphi, Y_a^\varphi) & & \end{array}$$

and δ vanishes on P . It remains to prove that $\tau(L, \chi) = 0$.

We have the following facts and remarks.

- (1) From the arguments of [FP14, Propositions 2.10, 2.12] (W_0 and W play the role of E_C and W_C , respectively), we have the following: if δ vanishes on P , then there exist an integer $l \geq k$ and a character $H_1(W^\varphi) \rightarrow \mathbb{Z}/q^l$, denoted by χ as an abuse of notation, which fits into the following diagram:

$$\begin{array}{ccccccc} \pi_1(M_L^\varphi) & \longrightarrow & H_1(M_L^\varphi) & \xrightarrow{\chi} & \mathbb{Z}/q^k & \xrightarrow{q^{l-k}} & \mathbb{Z}/q^l \\ \text{inc}_* \downarrow & & \text{inc}_* \downarrow & & & \nearrow \chi & \\ \pi_1(W^\varphi) & \longrightarrow & H_1(W^\varphi) & & & & \end{array}$$

- (2) Let $H_1(M_L) = \mathcal{H}'$ and $\phi': \pi_1(M_L) \rightarrow H_1(M_L)$ be the Hurewicz homomorphism. Define $\phi: \pi_1(M_L^\varphi) \rightarrow \mathcal{H}'$ be the restriction of ϕ' to the subgroup $\pi_1(M_L^\varphi)$. Let $\mathcal{H} = \text{Im } \phi$. Since $H_1(M_L) \cong H_1(W)$, ϕ' extends to $\pi_1(W)$. Therefore, we use $\phi': \pi_1(W) \rightarrow \mathcal{H}'$ and its restriction $\phi: \pi_1(W^\varphi) \rightarrow \mathcal{H}$ as an abuse of notation. Note that \mathcal{H}' is isomorphic to \mathbb{Z}^3 and \mathcal{H} is also isomorphic to \mathbb{Z}^3 as a finite index subgroup of \mathcal{H} .
- (3) By (1) and (2), we have $\chi \times \phi: \pi_1(W^\varphi) \rightarrow \mathbb{Z}/q^l \times \mathcal{H}$. If we write $\mathcal{K} = \mathbb{C}(\mathcal{H})$, then $H_*(M_L^\varphi; \mathcal{K})$, $H_*(W^\varphi; \mathcal{K})$, and $H_*(W^\varphi, M_L^\varphi; \mathcal{K})$ can be defined from

$$\mathbb{Z}[\pi_1(W^\varphi)] \xrightarrow{\chi \times \phi} \mathbb{Z}[\mathbb{Z}/q^l \times \mathcal{H}] = \mathbb{Z}[\mathbb{Z}/q^l][\mathcal{H}] \longrightarrow \mathbb{Q}(\xi_{q^l})(\mathcal{H}) \longrightarrow \mathbb{C}(\mathcal{H}) = \mathcal{K}.$$

- (4) By [FP14, Lemma 3.4], there is a 4-manifold W_χ bounded by $2q^l$ copies of a 3-torus T^3 , which is over $\mathbb{Z}/q^l \times \mathcal{H}$ as follows:

$$\begin{array}{ccc}
 \bigsqcup^{q^l} T^3 & & \\
 \downarrow & \searrow \chi \times \psi & \\
 W_\chi & \xrightarrow{\chi \times \psi} & K(\mathbb{Z}/q^l \times \mathcal{H}, 1) \\
 \uparrow & \nearrow \text{tr} \times \psi & \\
 \bigsqcup^{q^l} T^3 & &
 \end{array}$$

Here, tr denotes the trivial character $\pi_1(T^3) \rightarrow \mathbb{Z}/q^l$ and $\psi: \pi_1(T^3) \cong \mathcal{H}$. Furthermore, the intersection forms of W_χ over \mathbb{Q} -coefficient and \mathcal{K} -coefficients are Witt-trivial.

We can attach $\chi \times \phi: W^\varphi \rightarrow \mathbb{Z}/q^l \times \mathcal{H}$ and W_χ in (4) along $\chi \times \psi: T^3 \rightarrow \mathbb{Z}/q^l \times \mathcal{H}$ to obtain the cobordism $(W^\varphi \cup W_\chi, \chi \times \phi \cup \chi \times \psi)$ over $\mathbb{Z}/q^l \times \mathcal{H}$ between $(M_L^\varphi, \chi \times \phi)$ and $-(T^3, \text{tr} \times \psi)$. From Definition 4.1,

$$\tau(L, \chi) = (\lambda_{\mathcal{K}}(W^\varphi \cup W_\chi) - \mathcal{K} \otimes \lambda_{\mathbb{Q}}(W^\varphi \cup W_\chi)) \otimes 1 \in L^0(\mathcal{K}) \otimes_{\mathbb{Z}} \mathbb{Z}[1/q].$$

By (4), $[\lambda_{\mathbb{Q}}(W_\chi)] = 0 \in L^0(\mathbb{Q})$ and $[\lambda_{\mathcal{K}}(W_\chi)] = 0 \in L^0(\mathcal{K})$. In the following two claims we will prove that $[\lambda_{\mathbb{Q}}(W^\varphi)] = 0 \in L^0(\mathbb{Q})$ and $[\lambda_{\mathcal{K}}(W^\varphi)] = 0 \in L^0(\mathcal{K})$. By Novikov additivity, these will complete the proof of Theorem A.

Claim 1. $[\lambda_{\mathbb{Q}}(W^\varphi)] = 0 \in L^0(\mathbb{Q})$.

Proof of Claim 1. Applying relative Mayer-Vietoris (see [Hat02, page 152]), $H_i(W_0, X_J) \cong H_i(W, M_J)$ for all i and $J = L$ or H . (The other terms in the long exact sequence vanish because $H_*(X_H \times I, X_H) = 0$.) Similarly, $H_i(W_0^\varphi, X_J^\varphi) \cong H_i(W^\varphi, M_J^\varphi)$.

For brevity, let $A = \mathbb{Z}/p^i \oplus \mathbb{Z}/p^j$ and write $A = \{g_1, \dots, g_t\}$. Since $(W_0; X_L, X_H)$ is a 1.5-solvable cobordism, by Proposition 3.2 (1) and (2),

$$\beta_2(W^\varphi, M_L^\varphi) = \beta_2(W_0^\varphi, X_L^\varphi) = |A| \cdot \beta_2(W_0, X_L) = 2rt$$

and $\text{inc}_*: H_2(W_0^\varphi; \mathbb{Q}) \rightarrow H_2(W_0^\varphi, X_J^\varphi; \mathbb{Q})$ is surjective for $J = L, H$. Since $H_2(W_0^\varphi, X_J^\varphi; \mathbb{Q}) \cong H_2(W^\varphi, M_J^\varphi; \mathbb{Q})$, $\text{inc}_*: H_2(W^\varphi; \mathbb{Q}) \rightarrow H_2(W^\varphi, M_J^\varphi; \mathbb{Q})$ is surjective, too. Applying Proposition 3.2 (3), $H_i(W_0^\varphi, M_J^\varphi; \mathbb{Q}) = 0$ for $i = 0, 1$. From the homology long exact sequence of a pair (W^φ, M_J^φ) , this proves that $(W^\varphi, M_L^\varphi, M_H^\varphi)$ is an H_1 -cobordism over \mathbb{Q} -coefficients.

Recall $\partial W = M_L \sqcup -M_H$. For $X = \partial W^\varphi, M_L^\varphi$, and M_H^φ , let

$$I_X = \text{Im}(\text{inc}_*: H_2(X; \mathbb{Q}) \rightarrow H_2(W^\varphi; \mathbb{Q})).$$

For $J = L, H$, using the homology long exact sequences of pairs,

$$H_2(W^\varphi; \mathbb{Q})/I_{\partial W^\varphi} \cong H_2(W^\varphi; \mathbb{Q})/I_{M_J^\varphi} \cong H_2(W^\varphi, M_J^\varphi; \mathbb{Q})$$

whose rank is $2rt$. (Similar argument was used in the proof of [CK08, Proposition 2.6].) We remark that to prove the last isomorphism, we used the fact that $\text{inc}_*: H_1(M_J^\varphi; \mathbb{Q}) \rightarrow H_1(W^\varphi; \mathbb{Q})$ is an isomorphism for $J = L, H$.

Let $l_1, \dots, l_r, d_1, \dots, d_r$ be (generators of) 2-lagrangian and 1-duals in $H_2(W^\varphi; \mathbb{Z})$. From the (right) group action of A on $H_2(W^\varphi; \mathbb{Z})$, we define

$$l_{ij} = l_i \cdot g_j \text{ and } d_{kl} = d_k \cdot g_l \text{ for } 1 \leq i, k \leq r \text{ and } 1 \leq j, l \leq t.$$

The intersection pairing $\lambda_{\mathbb{Q}}(W^\varphi): H_2(W^\varphi; \mathbb{Q})/I_{\partial W^\varphi} \times H_2(W^\varphi; \mathbb{Q})/I_{\partial W^\varphi} \rightarrow \mathbb{Q}$ with respect to (the image of) $\{l_{ij}, d_{kl}\}$ is

$$\begin{pmatrix} 0 & I_{rt \times rt} \\ I_{rt \times rt} & X \end{pmatrix}$$

because $l_i \cdot d_k$ is the Kronecker delta δ_{ik} .

Let $L(\mathbb{Q}) \subset H_2(W^\varphi; \mathbb{Q})/I_{\partial W^\varphi}$ be the \mathbb{Q} -span of the image of $l_{ij} \otimes 1_{\mathbb{Q}}$. Then, $\lambda_{\mathbb{Q}}(W^\varphi)$ vanishes on $L(\mathbb{Q}) \times L(\mathbb{Q})$ and $\dim_{\mathbb{Q}} L(\mathbb{Q}) = \frac{1}{2} \dim_{\mathbb{Q}} (H_2(W^\varphi; \mathbb{Q})/I_{\partial W^\varphi}) = rt$. So, $[\lambda_{\mathbb{Q}}(W^\varphi)] = 0 \in L^0(\mathbb{Q})$. \square

By [FP14, Lemma 3.2], $H_*(M_J^\varphi; \mathcal{K}) = 0$ for $J = H$ or L . Therefore, the twisted intersection form

$$\lambda_{\mathcal{K}}(W^\varphi): H_2(W^\varphi; \mathcal{K}) \times H_2(W^\varphi; \mathcal{K}) \longrightarrow \mathcal{K}$$

is non-singular.

Claim 2. $[\lambda_{\mathcal{K}}(W^\varphi)] = 0 \in L^0(\mathcal{K})$.

Proof of Claim 2. Let $\alpha: \pi_1(W^\varphi) \xrightarrow{\chi} \mathbb{Z}/q^l \hookrightarrow \mathbb{C}^\times = \text{GL}(1, \mathbb{C})$ and $\alpha': \pi_1(W) \rightarrow \text{GL}(t, \mathbb{C})$ be the induced representation of α . Recall $\phi': \pi_1(W) \rightarrow \mathcal{H}'$ and $\phi: \pi_1(W^\varphi) \rightarrow \mathcal{H}$ in (2). Define $\Gamma := \text{Im}(\alpha \times \phi)$. There is a corresponding cover $(W^\Gamma, M_L^\Gamma) \rightarrow (W^\varphi, M_L^\varphi)$ where $\pi_1(W^\Gamma) = \text{Ker}(\alpha \times \phi)$. Recall $W^\varphi \rightarrow W$ is $\mathbb{Z}/p^i \oplus \mathbb{Z}/p^j$ -cover and $\alpha \times \phi: \pi_1(W^\varphi) \rightarrow \mathbb{C}^\times \times \mathcal{H}$. Since $\mathbb{Z}/p^i \oplus \mathbb{Z}/p^j$, \mathbb{C}^\times , and \mathcal{H} are abelian,

$$\pi_1(W)^{(2)} \leq \pi_1(W^\varphi)^{(1)} \leq \text{Ker}(\alpha \times \phi) = \pi_1(W^\Gamma)$$

Equivalently, there is a sequence of coverings:

$$W^{(2)} \longrightarrow W^\Gamma \longrightarrow W^\varphi \longrightarrow W$$

Since $\mathbb{Z}/q^l \hookrightarrow \mathbb{C}^\times$ is injective, $\text{Ker } \alpha = \text{Ker } \chi$, where $\alpha: \pi_1(W^\varphi) \xrightarrow{\chi} \mathbb{Z}/q^l \hookrightarrow \mathbb{C}^\times$. From this, $\Gamma \stackrel{\text{def}}{=} \text{Im}(\alpha \times \phi) = \text{Im}(\chi \times \phi)$. In particular, the ring homomorphism $\mathbb{Z}[\pi_1(W^\varphi)] \rightarrow \mathcal{K}$ in (3) factors through $\mathbb{Z}\Gamma$ and

$$C_*(W^\varphi; \mathcal{K}) \stackrel{\text{def}}{=} \mathcal{K} \otimes_{\mathbb{Z}[\pi_1(W^\varphi)]} C_*(W^\varphi; \mathbb{Z}[\pi_1(W^\varphi)]) = \mathcal{K} \otimes_{\mathbb{Z}\Gamma} C_*(W^\Gamma; \mathbb{Z}).$$

Choose 2-cycles $\{\tilde{l}_1, \dots, \tilde{l}_r\} \subset C_2(W^\Gamma; \mathbb{Z})$ which represent the image of (generators of) 2-lagrangian under the map induced by $W_0^{(2)} \rightarrow W^{(2)} \rightarrow W^\Gamma$. The covering map $W^\Gamma \rightarrow W^\varphi$ induces a surjection $\text{Cov}(W^\Gamma|W) \rightarrow \text{Cov}(W^\varphi|W) = \{g_1, \dots, g_t\}$. Choose a lift $\tilde{g}_j \in \text{Cov}(W^\Gamma|W)$ of g_j for each $j = 1, \dots, t$. From the right action of $\text{Cov}(W^\Gamma|W)$ on $C_2(W^\Gamma; \mathbb{Z})$, define

$$\tilde{l}_{ij} = \tilde{l}_i \cdot \tilde{g}_j \text{ for } 1 \leq i \leq r \text{ and } 1 \leq j \leq t.$$

Let

$$L(\mathcal{K}) \subset H_2(W^\varphi; \mathcal{K}) = H_2(\mathcal{K} \otimes_{\mathbb{Z}\Gamma} C_*(W^\Gamma; \mathbb{Z})),$$

be the \mathcal{K} -span of $\{[1_{\mathcal{K}} \otimes \tilde{l}_{ij}] \mid 1 \leq i \leq r, 1 \leq j \leq t\}$ in $H_2(W^\varphi; \mathcal{K})$. We remark that $L(\mathcal{K})$ does not depend on the choice of \tilde{g}_i . We claim that $L(\mathcal{K})$ is a lagrangian for the non-singular twisted intersection form $\lambda_{\mathcal{K}}(W^\varphi)$.

First, we prove $\lambda_{\mathcal{K}}$ vanishes on $L(\mathcal{K}) \times L(\mathcal{K})$. Since $\lambda_{\mathcal{K}}$ is \mathcal{K} -sesquilinear, the following is enough:

$$\lambda_{\mathcal{K}}([1_{\mathcal{K}} \otimes_{\mathbb{Z}\Gamma} \widetilde{l}_{ik}], [1_{\mathcal{K}} \otimes_{\mathbb{Z}\Gamma} \widetilde{l}_{jl}]) = \sum_{g \in \text{Cov}(W^\Gamma|W)} \lambda_{W^\Gamma}(\widetilde{l}_i, \widetilde{l}_j) g_l g g_k^{-1} = 0.$$

Now, we prove $\dim_{\mathcal{K}} L(\mathcal{K}) = \frac{1}{2} \dim_{\mathcal{K}} H_2(W^\varphi; \mathcal{K})$. Recall that $H_*(M_L; \mathcal{K}) = 0$ by [FP14, Lemma 3.2]. Therefore, $\text{inc}_*: H_2(W^\varphi; \mathcal{K}) \rightarrow H_2(W^\varphi, M_L^\varphi; \mathcal{K})$ is an isomorphism. Now, for simplicity, we abuse notation by regarding \widetilde{l}_{ij} as an element in $C_2(W^\Gamma, M_L^\Gamma; \mathbb{Z})$ and $L(\mathcal{K})$ as a subspace of $H_2(W^\varphi, M_L^\varphi; \mathcal{K})$.

Recall that $\{l_1, \dots, l_r\}$ is the chosen generators of 2-lagrangian in $H_2(W^\varphi; \mathbb{Z})$. Since the covering $W^\Gamma \rightarrow W$ sends \widetilde{g}_j to 1, the image of $\{[\widetilde{l}_{ij}] \in H_2(W^\Gamma; \mathbb{Z}) \mid 1 \leq i \leq r, 1 \leq j \leq t\}$ in $H_2(W, M_L; \mathbb{Z})$ (via covering induced map) is exactly $\{\pi(l_1), \dots, \pi(l_r)\}$ where $\pi: H_2(W^\varphi) \rightarrow H_2(W) \rightarrow H_2(W, M_L)$.

Since $(W_0; X_L, X_H)$ is a 1.5-solvable cobordism with $\beta_2(W_0, X_L) = 2r$, $H_2(W, M_L) \cong H_2(W_0, X_L)$ is a free abelian group of rank $2r$. Let

$$L(\mathbb{Z}/q) \subset H_2(W, M_L; \mathbb{Z}/q) \cong (\mathbb{Z}/q)^{2r}$$

be the \mathbb{Z}/q -span of $\{\pi(l_i) \otimes_{\mathbb{Z}} 1_{\mathbb{Z}/q}\}_{i=1}^r$. By the definition of 2-lagrangian, $\{\pi(l_1), \dots, \pi(l_r)\}$ generates a rank r -summand of $H_2(W, M_L) \cong \mathbb{Z}^{2r}$. In particular, from the universal coefficient theorem, $\dim_{\mathbb{Z}/q} L(\mathbb{Z}/q) = r$.

To apply Theorem 2.3, we fit our notations with those used in Section 2. Define $A = \mathbb{Z}/p^i \oplus \mathbb{Z}/p^j$, $G = \pi_1(W)$, $K = \pi_1(W^\varphi)$, $C_* = C_*(W, M_L; \mathbb{Z}[\pi_1(W)])$, $Q = \mathbb{C}$, $Q(\mathcal{H}) = \mathcal{K}$, $d = 1$, $\alpha \times \phi: \pi_1(W^\varphi) \rightarrow \mathbb{C}^\times \times \mathcal{H}$, and $\alpha' \times \phi': \pi_1(W) \rightarrow \text{GL}(t, \mathbb{C}) \times \mathcal{H}'$. (As a $\mathbb{Z}K$ -module, C_* is isomorphic to $C_*(W^\varphi, M_L^\varphi; \mathbb{Z}[\pi_1(W^\varphi)])$.) We remark that we assumed in Section 2 that $\alpha|_{\text{Ker } \phi}$ factors through a q -group for some prime q . This is automatically satisfied for $\alpha: \pi_1(W^\varphi) \xrightarrow{\chi} \mathbb{Z}/q^l \hookrightarrow \mathbb{C}^\times$.

With these notations, apply Theorem 2.3 for the case $I = \emptyset$ (that is, $M = \overline{M} = 0$) and $n = 0, 1$ to obtain

$$\dim_{\mathcal{K}} H_n(W^\varphi, M_J^\varphi; \mathcal{K}) \leq \dim_{\mathbb{Z}/q} H_n(W, M_J; \mathbb{Z}/q) = 0$$

for $n = 0, 1$ and $J = L$ or H . By duality and universal coefficient spectral sequence, $H_i(W^\varphi, M_L^\varphi; \mathcal{K}) = 0$ for $i = 3, 4$. From this,

$$\dim_{\mathcal{K}} H_2(W^\varphi, M_L^\varphi; \mathcal{K}) = \chi^{\mathcal{K}}(W^\varphi, M_L^\varphi) = \chi^{\mathbb{Q}}(W^\varphi, M_L^\varphi) = 2rt.$$

The last equality is from $\beta_2(W^\varphi, M_L^\varphi) = 2rt$ and $(W^\varphi; M_L^\varphi, M_H^\varphi)$ is an H_1 -cobordism over \mathbb{Q} -coefficient. These are proved in the proof of Claim 1.

Now, we apply Theorem 2.3 for the case $n = 2$, $I = \{i \mid 1 \leq i \leq r\}$ and x_i is a 2-cycle in C_* such that

$$[1_{\mathcal{K}} \otimes_{\mathbb{Z}\Gamma} \widetilde{l}_i] = [1_{\mathcal{K}} \otimes_{\mathbb{Z}K} x_i] \in H_2(\mathcal{K} \otimes_{\mathbb{Z}K} C_*) = H_2(W^\varphi, M_L^\varphi; \mathcal{K}) \text{ for } i = 1, \dots, r.$$

Recall $\widetilde{l}_{ij} = \widetilde{l}_i \cdot \widetilde{g}_j$, $\widetilde{g}_j \in \text{Cov}(W^\Gamma|W)$ is a lifting of $g_j \in \text{Cov}(W^\varphi|W)$. Since $\text{Cov}(W^\varphi|W)$ can be identified with the set of cosets of K in G , by the definition in Theorem 2.3,

$$M = \text{the } \mathcal{K}\text{-span of } \{[1_{\mathcal{K}} \otimes_{\mathbb{Z}\Gamma} \widetilde{l}_{ij}] \mid 1 \leq i \leq r, 1 \leq j \leq t\} = L(\mathcal{K}).$$

Similarly, by the definition in Theorem 2.3, \overline{M} is the \mathbb{Z}/q -span of $\{[1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} x_i]\}_{i=1}^r$. Since $\{[1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} x_i]\}_{i=1}^r = \{1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}} \pi(l_i)\}_{i=1}^r$,

$$\overline{M} = \text{the } \mathbb{Z}/q\text{-span of } \{1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}} \pi(l_i) \mid 1 \leq i \leq r\} = L(\mathbb{Z}/q).$$

From the conclusion of Theorem 2.3 for the above case, we have the following inequality

$$\dim_{\mathcal{K}} H_2(W^\varphi, M_L^\varphi; \mathcal{K}) - \dim_{\mathcal{K}} L(\mathcal{K}) \leq t \cdot (\dim_{\mathbb{Z}/q} H_2(W, M_L; \mathbb{Z}/q) - \dim_{\mathbb{Z}/q} L(\mathbb{Z}/q)).$$

That is,

$$\begin{aligned} \dim_{\mathcal{K}} L(\mathcal{K}) &\geq \dim_{\mathcal{K}} H_2(W, M_L; \mathcal{K}) - t \cdot (\dim_{\mathbb{Z}/q} H_2(W, M_L; \mathbb{Z}/q) - \dim_{\mathbb{Z}/q} L(\mathbb{Z}/q)) \\ &= 2rt - rt + rt = rt. \end{aligned}$$

On the other hand, $\dim_{\mathcal{K}} L(\mathcal{K}) \leq rt$ because $L(\mathcal{K})$ is the \mathcal{K} -span of rt elements. So, $\dim_{\mathcal{K}} L(\mathcal{K}) = rt = \frac{1}{2} \dim_{\mathcal{K}} H_2(W^\varphi; \mathcal{K})$ and $L(\mathcal{K})$ is a lagrangian of $\lambda_{\mathcal{K}}(W^\varphi)$. That is, $[\lambda_{\mathcal{K}}(W^\varphi)] = 0 \in L^0(\mathcal{K})$. \square

5. Solvable cobordism and abelian invariants of links

In this section, we study the abelian invariants of links (studied in [Kaw78] and [Hil12]) in the context of Whitney tower/grope concordance using h -solvable cobordism. Throughout this section, μ is the fixed natural number. Denote $\mathbb{Z}[t_1^\pm, \dots, t_\mu^\pm]$ by Λ_μ . The ring Λ_μ is endowed with the involution $-: t_i \mapsto t_i^{-1}$. Let S be the multiplicative set generated by $\{t_1 - 1, \dots, t_\mu - 1\}$. Denote the localization of Λ_μ with respect to S by $\Lambda_{\mu S}$. Let \mathcal{K} be the quotient field of Λ_μ .

5.1. Blanchfield form of μ -component links

Let L be a μ -component link and X_L be the link exterior of L . Let R be a unique factorization domain with an involution $-$ and quotient field K (our case is $R = \Lambda_{\mu S}, K = \mathcal{K}$). We recall the definition of the Witt group $W(K, R, -)$.

A *linking pairing over R* is a R -module M with a sesquilinear pairing

$$b: M \times M \longrightarrow K/R$$

such that for all $x, y, z \in M$ and $r \in R$

- (1) $b(x, y + z) = b(x, y) + b(x, z)$
- (2) $b(rx, y) = rb(x, y) = b(x, \bar{r}y)$
- (3) $b(x, y) = \overline{b(y, x)}$

(Here, the involution $-$ on K/R is induced from the involution on R .) We denote it by (M, b) or just b when M is clearly understood. A linking pairing (M, b) is *primitive* (*non-singular*) if the adjoint of b ,

$$\text{Ad}(b): M \longrightarrow \text{Hom}_R(M, K/R)$$

is an injection (an R -module isomorphism), respectively. The sum of linking pairings (M, b) and (M', b') is $(M \oplus M', b \oplus b')$. A pairing (M, b) is *neutral* if there is a submodule N of M such that

$$N = N^\perp = \{m \in M \mid b(n, m) = 0 \ \forall n \in N\}.$$

Two pairings (M, b) and (M', b') are *Witt equivalent* if there are neutral pairings (N, c) and (N', c') such that $(M, b) \oplus (N, c) \cong (M', b') \oplus (N', c')$. Then, the set of Witt equivalence classes of linking pairings over R with an involution $-$ is an abelian group, denoted by $W(K, R, -)$.

For a R -module M , following [Hil12, Chapter 3], we define the R -torsion submodule of M ,

$$tM = \{m \in M \mid rm = 0 \text{ for some } r \neq 0 \in R\} = \text{Ker}(M \longrightarrow M \otimes_R K),$$

the maximal pseudonull submodule of M ,

$$zM = \text{Ker}(tM \longrightarrow \text{Ext}_R^1(\text{Ext}_R^1(tM, R), R)),$$

and

$$\hat{t}M = tM/zM.$$

Note that a R -module M is called *pseudonull* if $M_{\mathfrak{p}} = 0$ for every height 1 prime ideal \mathfrak{p} of R .

From the Alexander duality, the Hurewicz map becomes $\pi_1(X_L) \rightarrow H_1(X_L) = \mathbb{Z}^\mu$. We have the following exact sequence

$$H_1(\partial X_L; \Lambda_\mu) \longrightarrow H_1(X_L; \Lambda_\mu) \longrightarrow H_1(X_L, \partial X_L; \Lambda_\mu) \longrightarrow H_0(\partial X_L; \Lambda_\mu)$$

whose external terms are $\prod_{i=1}^\mu (t_i - 1)$ -torsion (in particular, S -torsion) because \mathbb{Z}^μ -cover of ∂X_L is a disjoint union of $S^1 \times \mathbb{R}$ or $\mathbb{R} \times \mathbb{R}$. From this observation, by localizing the above sequence with respect to S , we obtain $H_1(X_L; \Lambda_{\mu S}) \cong H_1(X_L, \partial X_L; \Lambda_{\mu S})$. It follows from the (localized) Blanchfield duality [Bla57] (as in [Hil12, page 36]) that we have the following primitive linking pairing :

$$b_L : \hat{t}H_1(X_L; \Lambda_{\mu S}) \times \hat{t}H_1(X_L; \Lambda_{\mu S}) \longrightarrow \mathcal{K}/\Lambda_{\mu S}.$$

Here, to define b_L , we need the fact that $\mathcal{K}/\Lambda_{\mu S}$ contains no nontrivial pseudonull submodule, [Hil12, Theorem 3.9 (2)].

In this setting, Hillman [Hil12, Theorem 2.4] proved that $[b_L] \in W(\mathcal{K}, \Lambda_{\mu S}, -)$ is a concordance invariant of L . Here is our theorem which generalizes [Hil12, Theorem 2.4].

Theorem B. *Suppose L_0 and L_1 are μ -component links. If two link exteriors X_{L_0} and X_{L_1} are 1-solvable cobordant, then $[b_{L_0}] = [b_{L_1}] \in W(\mathcal{K}, \Lambda_{\mu S}, -)$. In particular, the conclusion holds if L_0 and L_1 are height 3 Whitney tower/grope concordant.*

Proof. Let W be a 1-solvable cobordism between X_{L_0} and X_{L_1} . Note that

$$\partial W = X_{L_0} \cup \mu(S^1 \times S^1 \times I) \cup -X_{L_1}$$

and $\mathbb{Z}^\mu = H_1(X_{L_i}) \xrightarrow{\text{inc}_*} H_1(W)$ is an isomorphism for $i = 0, 1$.

By the $(\Lambda_{\mu S}$ -coefficient) Mayer-Vietoris sequence of the triple $(\partial W, X_{L_0}, X_{L_1})$,

$$H_1(\partial W; \Lambda_{\mu S}) \cong H_1(X_{L_0}; \Lambda_{\mu S}) \oplus H_1(X_{L_1}; \Lambda_{\mu S}),$$

since $H_i(\mu(S^1 \times S^1 \times I); \Lambda_\mu)$ is S -torsion for $i = 0, 1$. From this, the (localized) Blanchfield form

$$b_{\partial W} : \hat{t}H_1(\partial W; \Lambda_{\mu S}) \times \hat{t}H_1(\partial W; \Lambda_{\mu S}) \longrightarrow \mathcal{K}/\Lambda_{\mu S}$$

is the direct sum $b_{L_0} \oplus (-b_{L_1})$. Therefore, it suffices to find a submodule Q of $\hat{t}H_1(\partial W; \Lambda_{\mu S})$ such that $Q = Q^\perp$.

By applying [Cha14, Theorem 4.13] to $n = 1$, $G = \mathbb{Z}^\mu$, $\phi : \pi_1(W) \rightarrow H_1(W) = \mathbb{Z}^\mu$, and $R = \mathbb{Z}$, we have the following Lemma.

Lemma 5.1. [Cha14, Theorem 4.13] *In the above situation,*

$$tH_2(W, \partial W; \Lambda_\mu) \longrightarrow tH_1(\partial W; \Lambda_\mu) \longrightarrow tH_1(W; \Lambda_\mu)$$

is exact.

Let $I_{\partial W}$ and I_W be $\Lambda_{\mu S}$ -coefficient intersection forms of ∂W and W , respectively. We have Blanchfield form,

$$b_W: \hat{t}H_1(W; \Lambda_{\mu S}) \times \hat{t}H_2(W, \partial W; \Lambda_{\mu S}) \longrightarrow \mathcal{K}/\Lambda_{\mu S}.$$

Let $P = \text{Im}(tH_2(W, \partial W; \Lambda_{\mu S}) \rightarrow \hat{t}H_1(\partial W; \Lambda_{\mu S}))$. Choose relative 2-cycles Q and R in $C_2(W, \partial W; \Lambda_{\mu S})$ representing the classes in $tH_2(W, \partial W; \Lambda_{\mu S})$. Denote the boundaries of Q and R by $q, r \in C_1(\partial W; \Lambda_{\mu S})$, respectively. The corresponding classes $[q], [r]$ in $\hat{t}H_1(\partial W; \Lambda_{\mu S})$ are actually in P . There exists $a \in \Lambda_{\mu S} - \{0\}$ such that $aq = \partial u$ for some $u \in C_2(\partial W; \Lambda_{\mu S})$. Then,

$$b_{\partial W}([q], [r]) = a^{-1}I_{\partial W}(u, r) = -a^{-1}I_W(i_*(u), R) = -b_W([\partial Q], [R]) \pmod{\Lambda_{\mu S}}.$$

(Here, $i_*: C_2(\partial W; \Lambda_{\mu S}) \rightarrow C_2(W; \Lambda_{\mu S})$.) Note that $[\partial Q] = 0 \in \hat{t}H_1(W; \Lambda_{\mu S})$. Therefore,

$$b_{\partial W}([q], [r]) = -b_W([\partial Q], [R]) = 0 \text{ for all } [q], [r] \in P.$$

This shows that $P \leq P^\perp$. Suppose that $x \in C_1(\partial W; \Lambda_{\mu S})$ represents a torsion class in $tH_1(\partial W; \Lambda_{\mu S})$ and $[x] \in P^\perp$. That is,

$$b_{\partial W}([x], [y]) = 0 \text{ for all } y = \partial Y, [Y] \in \hat{t}H_2(W, \partial W; \Lambda_{\mu S}).$$

So,

$$b_W([x], [Y]) = -b_{\partial W}([x], [y]) = 0 \text{ for all } [Y] \in \hat{t}H_2(W, \partial W; \Lambda_{\mu S}).$$

By Blanchfield duality for $(W, \partial W)$, the adjoint of b_W ,

$$\text{Ad}(b_W): \hat{t}H_1(W; \Lambda_{\mu S}) \longrightarrow \text{Hom}_{\Lambda_{\mu S}}(\hat{t}H_2(W, \partial W; \Lambda_{\mu S}), \mathcal{K}/\Lambda_{\mu S})$$

is injective. Therefore, $[x] = 0 \in \hat{t}H_1(W; \Lambda_{\mu S})$ or x represents a homology class in $zH_1(W; \Lambda_{\mu S})$. This shows that P^\perp/P is a pseudonull $\Lambda_{\mu S}$ -module. We claim that $P^{\perp\perp} = P^\perp$. The inclusion maps $P \hookrightarrow P^\perp$ and $P^\perp \hookrightarrow \hat{t}H_1(\partial W; \Lambda_{\mu S})$ induce two vertical maps in the following diagram (here, the horizontal map is $\text{Ad}(b_W)$):

$$\begin{array}{ccc} \hat{t}H_1(\partial W; \Lambda_{\mu S}) & \xrightarrow{\quad} & \text{Hom}_{\Lambda_{\mu S}}(\hat{t}H_1(\partial W; \Lambda_{\mu S}), \mathcal{K}/\Lambda_{\mu S}) \\ & \searrow i & \downarrow \\ & & \text{Hom}_{\Lambda_{\mu S}}(P^\perp, \mathcal{K}/\Lambda_{\mu S}) \\ & \searrow j \circ i & \downarrow j \\ & & \text{Hom}_{\Lambda_{\mu S}}(P, \mathcal{K}/\Lambda_{\mu S}) \end{array}$$

By definition, $P^{\perp\perp} = \text{Ker } i$ and $P^\perp = \text{Ker}(j \circ i)$. By applying $\text{Hom}_{\Lambda_{\mu S}}(-, \mathcal{K}/\Lambda_{\mu S})$ to $0 \rightarrow P \rightarrow P^\perp \rightarrow P^\perp/P \rightarrow 0$, we obtain that

$$0 \longrightarrow \text{Hom}_{\Lambda_{\mu S}}(P^\perp/P, \mathcal{K}/\Lambda_{\mu S}) \longrightarrow \text{Hom}_{\Lambda_{\mu S}}(P^\perp, \mathcal{K}/\Lambda_{\mu S}) \xrightarrow{j} \text{Hom}_{\Lambda_{\mu S}}(P, \mathcal{K}/\Lambda_{\mu S})$$

is exact. That is, $\text{Ker } j \cong \text{Hom}_{\Lambda_{\mu S}}(P^\perp/P, \mathcal{K}/\Lambda_{\mu S})$.

From the short exact sequence $0 \rightarrow \Lambda_{\mu S} \rightarrow \mathcal{K} \rightarrow \mathcal{K}/\Lambda_{\mu S} \rightarrow 0$, the following is exact :

$$\text{Hom}_{\Lambda_{\mu S}}(P^\perp/P, \mathcal{K}) \longrightarrow \text{Hom}_{\Lambda_{\mu S}}(P^\perp/P, \mathcal{K}/\Lambda_{\mu S}) \longrightarrow \text{Ext}_{\Lambda_{\mu S}}^1(P^\perp/P, \Lambda_{\mu S}).$$

Since P^\perp/P is $\Lambda_{\mu S}$ -torsion and \mathcal{K} is $\Lambda_{\mu S}$ -torsion free, $\text{Hom}_{\Lambda_{\mu S}}(P^\perp/P, \mathcal{K}) = 0$. Also, P^\perp/P is a pseudonull $\Lambda_{\mu S}$ -module implies that $\text{Ext}_{\Lambda_{\mu S}}^1(P^\perp/P, \Lambda_{\mu S}) = 0$. (By Theorem 3.9 of

[Hil12], for a unique factorization domain R and R -module M , M is pseudonull if and only if $\text{Hom}_R(M, R) = 0$ and $\text{Ext}_R^1(M, R) = 0$.) Therefore,

$$\text{Ker } j \cong \text{Hom}_{\Lambda_{\mu S}}(P^\perp/P, \mathcal{K}/\Lambda_{\mu S}) = 0.$$

From the kernel-cokernel exact sequence,

$$0 \longrightarrow \text{Ker } i \longrightarrow \text{Ker}(j \circ i) \longrightarrow \text{Ker } j = 0$$

is exact. This shows that $P^\perp = \text{Ker}(j \circ i) \cong \text{Ker } i = P^{\perp\perp}$ and completes the proof. \square

5.2. Multivariable Alexander polynomial of links

In this subsection, we prove Theorem C which generalizes [Kaw78, Theorems A, B] concerning the Fox-Milnor condition for the Alexander polynomial of links.

First, we recall some definitions of [Kaw78]. Since Λ_μ is Nötherian, for a finitely generated Λ_μ -module M , we can choose a presentation matrix P of M from an exact sequence $\Lambda_\mu^m \xrightarrow{P} \Lambda_\mu^n \rightarrow M \rightarrow 0$. Moreover, for all k , one can choose a $m \times n$ presentation matrix P with $n > k$ and $m \geq n - k$. In this situation, define the k -th Alexander polynomial of M , denoted by $\Delta_k(M)$, to be the greatest common divisor of the size $(n - k) \times (n - k)$ minors of P . (It is well-known that $\Delta_k(M)$ is well-defined up to a unit of Λ_μ which is proved in [CF77].)

Remark 5.2. (1) From [Bla57, Theorem 4.10], if $d = \text{rank}_{\Lambda_\mu} M$, then $\Delta_d(M) = \Delta_0(tM)$.
 (2) From [Kaw78, Lemma 2.4], if $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is a short exact sequence of Λ_μ -torsion modules, then $\Delta_0(M) = \Delta_0(M')\Delta_0(M'')$.
 (3) By definition, for a nonzero Λ_μ -torsion module M , $\Delta_0(M) \neq 0$.

Recall that L is a μ -component link in S^3 and the meridian map is $\pi_1(X_L) \rightarrow \mathbb{Z}^\mu$. We define the torsion Alexander polynomial of L by $\Delta_L^T := \Delta_0(tH_1(X_L; \Lambda_\mu))$. Now we state our theorem.

Theorem C. *Suppose L_0 and L_1 are μ -component links. If two link exteriors X_{L_0} and X_{L_1} are 1-solvable cobordant, then*

- (1) $\text{rank}_{\Lambda_\mu} H_1(X_{L_0}; \Lambda_\mu) = \text{rank}_{\Lambda_\mu} H_1(X_{L_1}; \Lambda_\mu)$ and
- (2) $\Delta_{L_0}^T f_0 \overline{f_0} \doteq \Delta_{L_1}^T f_1 \overline{f_1}$ for some $f_i(t_1, \dots, t_\mu) \in \Lambda_\mu$, $i = 0, 1$ with $|f_i(1, \dots, 1)| = 1$.

In particular, the conclusion holds if L_0 and L_1 are height 3 Whitney tower/grope concordant.

To prove Theorem C, we need to prove the following generalization of [Kaw78, Lemma 2.1].

Lemma 5.3. *Let X be a finite connected CW-complex with an epimorphism*

$$\gamma: \pi_1(X) \longrightarrow \mathbb{Z}^\mu$$

Let X_0 be a subcomplex of X . For some fixed k , if $H_k(X, X_0; \mathbb{Z}) = \mathbb{Z}^l$ and

$$\text{rank}_{\Lambda_\mu} H_k(X, X_0; \Lambda_\mu) = l$$

then the l -th Alexander polynomial $A = \Delta_l(H_k(X, X_0; \Lambda_\mu)) = \Delta_0(tH_k(X, X_0; \Lambda_\mu))$ satisfies $|A(1, \dots, 1)| = 1$.

Remark. For the $l = 0$ case ([Kaw78, Lemma 2.1]), we only need to assume $H_k(X, X_0; \mathbb{Z}) = 0$ because from our proof, we can deduce

$$\text{rank}_{\Lambda_\mu} H_k(X, X_0; \Lambda_\mu) \leq 0.$$

In this sense, Lemma 5.3 is a generalization of [Kaw78, Lemma 2.1].

Proof of Lemma 5.3. Since X_0 is a subcomplex of X , for all q , we fix a basis for the q -th (cellular) chain complex $C_q(X, X_0; \mathbb{Z}) \cong \mathbb{Z}^{s_q}$. By lifting each element in the chosen bases, we also fix a Λ_μ -basis for the $C_q(X, X_0; \Lambda_\mu)$ for all q . With these chosen bases, we can write $\partial_q: C_q(X, X_0; \Lambda_\mu) \rightarrow C_{q-1}(X, X_0; \Lambda_\mu)$ as a matrix (α_{ij}^q) , $\alpha_{ij}^q \in \Lambda_\mu$.

With respect to the chosen basis of $C_*(X, X_0; \mathbb{Z})$, $\partial_q: C_q(X, X_0; \mathbb{Z}) \rightarrow C_{q-1}(X, X_0; \mathbb{Z})$ is represented by the integral matrix $(\alpha_{ij}^q(1, \dots, 1))$. Let $\tilde{r}_q = \text{rank}(\alpha_{ij}^q)$, $r_q = \text{rank}(\alpha_{ij}^q(1, \dots, 1))$. Then, $r_q \leq \tilde{r}_q$. Since $H_k(X, X_0; \mathbb{Z}) = \mathbb{Z}^l$,

$$l = \text{rank}_{\mathbb{Z}} \text{Ker } \partial_k - \text{rank}_{\mathbb{Z}} \text{Im } \partial_{k+1} = s_k - r_k - r_{k+1}.$$

Similarly, from $\text{rank}_{\Lambda_\mu} H_k(X, X_0; \Lambda_\mu) = l$,

$$l = s_k - \tilde{r}_k - \widetilde{r_{k+1}}$$

Since $r_q \leq \tilde{r}_q$ for all q ,

$$l = s_k - \tilde{r}_k - \widetilde{r_{k+1}} \leq s_k - r_k - r_{k+1} = l$$

which implies that $r_k = \tilde{r}_k$, $r_{k+1} = \widetilde{r_{k+1}}$.

Since $\text{Coker } \partial_{k+1} = C_k(X, X_0; \Lambda_\mu) / \text{Im } \partial_{k+1}$ and $\text{Im } \partial_k \cong C_k(X, X_0; \Lambda_\mu) / \text{Ker } \partial_k$, we have the short exact sequence

$$0 \longrightarrow H_k(X, X_0; \Lambda_\mu) \longrightarrow \text{Coker } \partial_{k+1} \longrightarrow \text{Im } \partial_k \longrightarrow 0$$

As a submodule of a free module, $\text{Im } \partial_k$ is a Λ_μ -torsion free module of rank $\tilde{r}_k = r_k$. Then, $tH_k(X, X_0; \Lambda_\mu) = t \text{Coker } \partial_{k+1}$ and $\dim_{\mathcal{K}} \text{Coker } \partial_{k+1} \otimes_{\Lambda_\mu} \mathcal{K} = l + r_k$.

$$\Delta_l(H_k(X, X_0; \Lambda_\mu)) = \Delta_0(tH_k(X, X_0; \Lambda_\mu)) = \Delta_0(t \text{Coker } \partial_{k+1}) = \Delta_{l+r_k}(\text{Coker } \partial_{k+1})$$

The first and last inequality follows from Remark 5.2 (1). Similarly, we have the short exact sequence,

$$0 \longrightarrow H_k(X, X_0; \mathbb{Z}) \longrightarrow \text{Coker } \partial_{k+1}^{\mathbb{Z}} \longrightarrow \text{Im } \partial_k^{\mathbb{Z}} \longrightarrow 0.$$

(Here, to avoid the confusion, we denote the differential on $C_*(X, X_0; \mathbb{Z})$ by $\partial_*^{\mathbb{Z}}$.) Since \mathbb{Z} is a PID, every submodule of finitely generated free \mathbb{Z} -module is free. So, $\text{Im } \partial_k^{\mathbb{Z}}$ is isomorphic to \mathbb{Z}^{r_k} . Therefore,

$$\text{Coker } \partial_{k+1}^{\mathbb{Z}} = H_k(X, X_0; \mathbb{Z}) \oplus \mathbb{Z}^{r_k} = \mathbb{Z}^{l+r_k}$$

(Here, we used the assumption that $H_k(X, X_0; \mathbb{Z}) = \mathbb{Z}^l$.) Note that the matrices (α_{ij}^{k+1}) and $(\alpha_{ij}^{k+1}(1, \dots, 1))$ are presentation matrices of $\text{Coker } \partial_{k+1}$ and $\text{Coker } \partial_{k+1}^{\mathbb{Z}}$, respectively. Therefore,

$$|\Delta_l(H_k(X, X_0; \Lambda_\mu))(1, \dots, 1)| = |\Delta_{l+r_k}(\text{Coker } \partial_{k+1})(1, \dots, 1)| = 1.$$

This completes the proof. \square

Proof of Theorem C. Let W be a 1-solvable cobordism between X_{L_0} and X_{L_1} . In particular, the inclusion induces $\mathbb{Z}^\mu = H_1(X_{L_0}) \cong H_1(W)$ and $H_1(W, X_{L_0}) = H_1(W, X_{L_1}) = 0$. By Poincaré duality and universal coefficient theorem,

$$H_2(W, X_{L_0}) \cong H^2(W, X_{L_1}) \cong \text{Hom}_{\mathbb{Z}}(H_2(W, X_{L_1}), \mathbb{Z}) = \mathbb{Z}^{2r}$$

(Since W is a 1-solvable cobordism between X_{L_0} and X_{L_1} , $\text{rank}_{\mathbb{Z}} H_2(W, X_{L_1})$ is even.) Let $C_* = C_*(W, X_{L_0}; \Lambda_\mu)$. Then,

$$H_i(C_* \otimes_{\Lambda_\mu} \mathbb{Z}) = H_i(W, X_{L_0}; \mathbb{Z}) = 0 \text{ for } i = 0, 1.$$

Since $\Lambda_\mu = \mathbb{Z}[\mathbb{Z}^\mu]$ and \mathbb{Z}^μ is a poly-torsion-free-abelian-group, by [COT03, Proposition 2.10],

$$H_i(C_* \otimes_{\Lambda_\mu} \mathcal{K}) = H_i(W, X_{L_0}; \Lambda_\mu) \otimes_{\Lambda_\mu} \mathcal{K} = 0 \text{ for } i = 0, 1.$$

Similarly, $H_i(W, X_{L_1}; \Lambda_\mu) \otimes_{\Lambda_\mu} \mathcal{K} = 0$ for $i = 0, 1$. From duality and universal coefficient spectral sequence, $H_i(W, X_{L_0}; \Lambda_\mu) \otimes_{\Lambda_\mu} \mathcal{K} = 0$ for $i = 3, 4$. So,

$$\text{rank}_{\Lambda_\mu} H_2(W, X_{L_i}; \Lambda_\mu) = \chi(C_*) = \chi(C_*(W, X_{L_i}; \mathbb{Z})) = \text{rank}_{\mathbb{Z}} H_2(W, X_{L_i}; \mathbb{Z}) = 2r$$

for $i = 0, 1$. As in Lemma 5.1, the existence of 1-lagrangians and 1-duals implies that the following is exact for $i = 0, 1$:

$$tH_2(W, X_{L_i}; \Lambda_\mu) \longrightarrow H_1(X_{L_i}; \Lambda_\mu) \longrightarrow H_1(W; \Lambda_\mu) \longrightarrow tH_1(W, X_{L_i}; \Lambda_\mu).$$

(Note that $H_1(W, X_{L_i}; \Lambda_\mu) = tH_1(W, X_{L_i}; \Lambda_\mu)$ for $i = 0, 1$.) In particular, (1) is proved because

$$\text{rank}_{\Lambda_\mu} H_1(X_{L_0}; \Lambda_\mu) = \text{rank}_{\Lambda_\mu} H_1(W; \Lambda_\mu) = \text{rank}_{\Lambda_\mu} H_1(X_{L_1}; \Lambda_\mu).$$

The following is also exact for $i = 0, 1$:

$$tH_2(W, X_{L_i}; \Lambda_\mu) \longrightarrow tH_1(X_{L_i}; \Lambda_\mu) \longrightarrow tH_1(W; \Lambda_\mu) \longrightarrow tH_1(W, X_{L_i}; \Lambda_\mu).$$

Now, fix i . Denote the 0-th Alexander polynomial of these modules and $tH_1(\partial W; \Lambda_\mu)$ by

$$\Delta_2, \Delta_{L_i}^T, \Delta_W, \Delta_1, \text{ and } \Delta_{\partial W}$$

respectively. (Of course, Δ_2 and Δ_1 depend on i .)

Note that $H_1(W) \cong \mathbb{Z}^\mu$ and $H_2(W, X_{L_i}; \mathbb{Z}) \cong \mathbb{Z}^{2r}$, so $\text{rank}_{\Lambda_\mu} H_2(W, X_{L_i}; \Lambda_\mu) = 2r$. Applying Lemma 5.3 to $(X, X_0) = (W, X_{L_i})$ for the cases $(k, l) = (2, 2r)$ and $(1, 0)$, we obtain $|\Delta_2(1, \dots, 1)| = |\Delta_1(1, \dots, 1)| = 1$. Using Remark 5.2 (2), $\Delta_{L_i}^T g_i \doteq \Delta_W g'_i$ for some $g_i, g'_i \in \Lambda_\mu$ with $|g_i(1, \dots, 1)| = |g'_i(1, \dots, 1)| = 1$ for $i = 0, 1$. In particular, $\Delta_{L_0}^T g \doteq \Delta_{L_1}^T g'$ for some $g, g' \in \Lambda_\mu$ with $|g(1, \dots, 1)| = |g'(1, \dots, 1)| = 1$.

Since Λ_μ is a unique factorization domain, we can split $\Delta_{L_i}^T = u_i v_i$ and $\Delta_{\partial W} = uv$ uniquely (up to units of Λ_μ) so that v_0, v_1, v consist of all irreducible factors $f \in \Lambda_\mu$ with $|f(1, \dots, 1)| \neq 1$ in $\Delta_{L_0}^T, \Delta_{L_1}^T, \Delta_{\partial W}$. From $\Delta_{L_0}^T g \doteq \Delta_{L_1}^T g'$ and $|g(1, \dots, 1)| = |g'(1, \dots, 1)| = 1$, $v_0 \doteq v_1$.

From the Mayer-Vietoris sequence, the following is exact:

$$tH_1(\partial X_{L_0}; \Lambda_\mu) \longrightarrow tH_1(X_{L_0}; \Lambda_\mu) \oplus tH_1(X_{L_1}; \Lambda_\mu) \longrightarrow tH_1(\partial W; \Lambda_\mu) \longrightarrow tH_0(\partial X_{L_0}; \Lambda_\mu).$$

The extreme terms are $\prod_{i=1}^\mu (t_i - 1)$ -torsion. It follows that $\Delta_{\partial W} \lambda \doteq \Delta_{L_0}^T \Delta_{L_1}^T \lambda'$ for some factors

λ, λ' of $\prod_{i=1}^\mu (t_i - 1)$. By the reciprocity of Blanchfield pairing [Bla57], $\Delta_{L_i}^T \doteq \overline{\Delta_{L_i}^T}$ for $i = 0, 1$.

Now, we have

$$u \doteq u_0 u_1 \doteq \overline{u_0} u_1.$$

By Theorem B, we proved that the Blanchfield form of ∂W is neutral, which implies that $\Delta_{\partial W} = h \bar{h}$ for some $h \in \Lambda_\mu$ by [Hil12, Theorem 3.27]. In particular,

$$u \doteq f \bar{f} \text{ for some } f \in \Lambda_\mu \text{ with } |f(1, \dots, 1)| = 1.$$

Combining all these observations,

$$\Delta_{L_0}^T f \bar{f} \doteq u_0 v_0 u = u_0 \overline{u_0} u_1 v_0 \doteq u_0 \overline{u_0} u_1 v_1 \doteq \Delta_{L_1}^T u_0 \overline{u_0}.$$

Here, f and u_0 satisfy the conditions $|f(1, \dots, 1)| = 1$, $|u_0(1, \dots, 1)| = 1$. This completes the proof. \square

Remark. It should be noted that Theorem C is not a direct consequence of Theorem B. From Theorem B without Lemma 5.3, one may deduce that if X_{L_0} and X_{L_1} are 1-solvable cobordant, then

$$(1) \text{ rank}_{\Lambda_\mu} H_1(X_{L_0}; \Lambda_\mu) = \text{rank}_{\Lambda_\mu} H_1(X_{L_1}; \Lambda_\mu) \text{ and}$$

$$(2) \Delta_{L_0}^T f_0 \overline{f_0} \doteq \Delta_{L_1}^T f_1 \overline{f_1}$$

for some $f_0, f_1 \in \Lambda_{\mu S} - \{0\}$. Lemma 5.3 is crucial to obtain the stronger conclusion that we can choose $f_0, f_1 \in \Lambda_\mu$ such that $|f_0(1, \dots, 1)| = |f_1(1, \dots, 1)| = 1$.

Finally, we mention what can be deduced from Theorems B and C for the special case of 2-component links with linking number 1. Note that by the work of Levine [Lev82], the Blanchfield form (without localization) $b_L: tH_1(X_L; \Lambda_2) \times tH_1(X_L; \Lambda_2) \rightarrow \mathcal{K}/\Lambda_2$ is non-singular.

Corollary D. *Suppose L is a 2-component link with linking number 1. If X_L and X_H are 1-solvable cobordant, then*

- (1) $[b_L] = 0 \in W(\mathcal{K}, \Lambda_2, -)$,
- (2) $\beta(L) = 0$,
- (3) $\Delta_0(L) \doteq f\overline{f}$ for some $f \in \Lambda_2$ such that $|f(1, 1)| = 1$.

In particular, the conclusion holds if L and H are height 3 Whitney tower/grope concordant.

Proof. Let L be a 2-component link with linking number 1. Assume that X_L and X_H are 1-solvable cobordant. Since $X_H = S^1 \times S^1 \times I$ and the $\mathbb{Z} \oplus \mathbb{Z}$ cover of X_H is $\mathbb{R} \times \mathbb{R} \times I$, $[b_H] = 0, \beta(H) = 0$ and $\Delta_0(H) = 1$. This shows (1) and (2). With the notation in the proof of Theorem C (applied to $L_0 = H$ and $L_1 = L$), $u_0 = 1$ and

$$\Delta_0(H)f\overline{f} \doteq \Delta_0(L)u_0\overline{u_0}$$

for some $f \in \Lambda_2$ such that $|f(1, 1)| = 1$. Since $\Delta_0(H) = 1$ and $u_0 = 1$, $\Delta_0(L) = f\overline{f}$ for some $f \in \Lambda_2$ such that $|f(1, 1)| = 1$. This completes the proof of (3). \square

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